Open Stability Problems

Sergey Foss

Heriot-Watt University, Edinburgh, U.K.

1 Stability of a greedy server

There are two continuous state space models where the stability conjecture is obvious, but nobody is able to verify it. In both models, the driving algorithm contains a "locally optimal" ("greedy") element. It looks like none of existing stability methods is working here.

Model 1. A single server is located on the circle. Particles arrive in a Poisson stream of rate λ and are uniformly distributed (as material points) on the circle (people say that there is a "Poisson rain" of particles). It takes a single unit of time to serve a particle. After any service, the particle disappears, the server chooses to serve next the *closest* particle and moves to it with a (positive finite) constant speed (ignoring new arrivals), serves it during another unit of time, then chooses the next closest particle and moves to it, etc.

The conjecture is: this model is stable for any $\lambda < 1$. A plausible "proof" might be as follows: if the number of requests is very large, then the server is busy with service almost all the time (with a service speed close to one), and then we may apply, say, fluid approximation ideas to deduce the stability.

This model and this conjecture are known already for more than 20 years, see [6], but nobody could succeed with obtaining either a proof or a counter-example here. The key problem is the continuity of the state space, and there are several results (see, e.g., [11] for further details) with the proof of a similar hypothesis for models with a finite state space (for instance, you may replace the continuous circle by a finite lattice on it). If the server uses any "state-independent" algorithm for moving (say, always walks in the left direction or chooses the next direction with probability $1/2$ independently of everything else), then it is easy to verify the conjecture using the ideas explained above.

Model 2. Again, there is a circle, but no any server and service this time. There are two independent Poisson streams/rains, of "black" and of "white" particles, with rates λ and 1, respectively. Black particles arrive at the circle and stop there, and white particles only pass the circle (here "pass" means "arrive and immediately disappear"). There is given a distance $\varepsilon > 0$. When a white particle passes the circle at some point, it observes all blacks in the ε-neighbourhood and takes (deletes) the one which is the closest to the white particle (if there is any).

The natural conjecture is: the stability should be guaranteed by condition $\lambda < 1$, independently of the circle length and the number ε. But the problem is open too. Again, there exist simple proofs for stability if the model is modified: if either the continuous state space (the circle) is replaces by a finite set, or the greedy mechanism is replaced by any state-independent mechanism (for instance, if a white particle takes one of blacks from the neighbourhood "at random", with equal probabilities).

2 Is there a coupling-convergence in two-server queue?

Consider a stable 2-server queue with stationary and ergodic driving sequence $\{\sigma_n, t_n\}$ of service and inter-arrival times, which starts from the empty state. Consider the Kiefer-Wolfowitz vectors $\mathbf{W}_0 = (0,0)$ and, for $n = 0, 1, \ldots$,

$$\mathbf{W}_{n+1} = R(\mathbf{W}_n + \mathbf{e}_1\sigma_n - \mathbf{i}t_n)^+$$

where $\mathbf{e}_1 = (1,0)$, $\mathbf{i} = (1,1)$, $R(x_1, x_2) = (x_{(1)}, x_{(2)})$ is the non-decreasing ordering, and $(x_1, x_2)^+ = (\max(0, x_1), \max(0, x_2))$.

Since $(0,0) \le \theta^{-1}\mathbf{W}_1 \le \theta^{-2}\mathbf{W}_2 \le \ldots$ a.s.,

$$\mathbf{W}_{n+1} \ge_{st} \mathbf{W}_n,$$

for any n. The stability condition is $\mathbf{E}\sigma_1 < 2\mathbf{E}t_1$. Under this condition, $\mathbf{W}_n$ converge weakly to the stationary vector.

One can show that, under various assumptions, the convergence holds in the total variation norm. Equivalently, the sequence $\theta^{-n}\mathbf{W}_n$ coupling-converges to the limit. The question is: is this always true, without any further assumptions ?

3 Does there exist an example where the null-recurrence occurs on the set of parameters of a positive Lebesgue measure ?

Consider the example of two-server-two-station model (see [10]).

Remark. I plan to publish a special issue of 'Queueing Systems" on open problems. Prof Onno Boxma (Editor-in-Chief) has already agreed with this idea, and there is a team of colleagues who volunteered to help me.

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