1 Appendix: the u.o.c. convergence

Let $f_n : \mathbf{R} \to \mathbf{R}$, $n = 1, 2, \ldots$, and $f : \mathbf{R} \to \mathbf{R}$ be any functions.

Definition 1. *$f_n \overset{\text{u.o.c.}}{\to} f$ as $n \to \infty$, if, for any compact (bounded and closed) set K,*

$$\sup_{x \in K} |f_n(x) - f(x)| \to 0 \text{ as } n \to \infty.$$

For any f and for any $t > 0$, define a norm $\|f\|_t$:

$$\|f\|_t = \sup_{|x| \le t} |f(x)|,$$

and a norm $\|f\|$:

$$\|f\| = \sum_{m=1}^{\infty} 2^{-m} \cdot \frac{\|f\|_m}{1 + \|f\|_m}.$$

Lemma 1. $f_n \overset{\text{u.o.c.}}{\to} f \iff \|f_n - f\| \to 0$ *as* $n \to \infty$.

Indeed,
($\Rightarrow$). $\forall\, \varepsilon > 0$, choose m_0:

$$\sum_{m=m_0+1}^{\infty} 2^{-m} \equiv 2^{-m_0} \le \frac{\varepsilon}{2}$$

Then choose n_0:

$$\|f_n(x) - f(x)\|_{m_0} \le \frac{\varepsilon}{2m_0} \quad \forall\, n \ge n_0.$$

$$\models\!\Rightarrow \|f_n - f\| \le \sum_{m=1}^{m_0} \|f_n - f\|_m + 2^{-m_0} \le m_0 \cdot \frac{\varepsilon}{2m_0} + \frac{\varepsilon}{2} = \varepsilon.$$

($\Leftarrow$). Let K be a compact non-empty set. Take $m_0 \gg 1$: $K \subseteq [-m_0, m_0]$. Set $g_n = f_n - f$. Then

$$\|f_n - f\| \to 0 \models\!\Rightarrow 2^{-m_0} \cdot \frac{\|g_n\|_{m_0}}{1 + \|g_n\|_{m_0}} \to 0 \models\!\Rightarrow \|g_n\|_{m_0} \to 0 \models\!\Rightarrow \sup_{x \in K} |g_n(x)| \to 0.$$

□

Remark 1. *For $\|f\|$ to be finite, all $\|f\|_m$ have to be finite. This is always the case if f is, say, continuous or monotone.*

Remark 2. *If f is defined on a measurable subset $B \subset \mathbf{R}$ only, we let $f(x) = 0$ $\forall\, x \notin B$.*

Remark 3. *Any (right- and/or left-) continuous function is determined by its values on a dense subset (e.g., by values at all rational points).*

Lemma 2. *Let all f_n, $n = 1, 2, \dots$ be non-decreasing functions and f a continuous function.*
Then the following are equivalent:

(a) $f_n \overset{\text{u.o.c.}}{\to} f$;

(b) *for any dense subset B of the real line, $f_n(x) \to f(x)$, $\forall\, x \in B$.*

Proof.
($\Leftarrow$ Only!). Take any m and prove that $\|f_n - f\|_m \to 0$. Assume, by the contrary, that

$$\limsup_{n\to\infty} \|f_n - f\|_m = C > 0 \qquad \text{(Note: } C < \infty!)$$

$\models\!\Longrightarrow$ $\exists$ a subsequence $\{n_l\}$: $\|f_{n_l} - f\|_m \to C$,
$\models\!\Longrightarrow$ since $[-m, m]$ is a compact set, $\exists$ a sequence $\{x_l\} \to x \in [-m, m]$ such that $|f_{n_l}(x_l) - f(x_l)| \geq C/2$, $\forall\, l$, $x_l \in [-m, m]$.
Choose $\delta > 0$: (i) $\delta \in B$;
(ii) $|f(x+\delta) - f(x)| \leq C/4$.
Then

$$f_{n_l}(x_l) - f(x_l) = f_{n_l}(x_l) - f_{n_l}(x+\delta) + f_{n_l}(x+\delta) - f(x+\delta) + f(x+\delta) - f(x) + f(x) - f(x_l)$$

and

$$\limsup_{l\to\infty} (f_{n_l}(x_l) - f(x_l)) \leq 0 + 0 + C/4 + 0 = C/4.$$

Similarly, choose $\tilde{\delta} > 0$: (i) $\tilde{\delta} \in B$;
(ii) $f(x - \tilde{\delta}) - f(x) \leq -C/4$.
Then

$$\liminf_{l\to\infty} (f_{n_l}(x_l) - f(x_l)) \geq -C/4 \text{ — we arrive at a contradiction!}$$

□

Lemma 3. *(The "triangular" scheme).*
Let $D, d > 0$ and $\{f_n\}$: $\sup_{|x|\leq d} |f_n(x)| \leq D\ \forall\, n$.
Then $\exists$ a subsequence $\{n_r\}$: $\forall$ rational x,

$$\exists \textit{ a limit } \lim_{r\to\infty} f_{n_r}(x) := f(x).$$

Proof. Number all rational points in $[-d, d]$ in an arbitrary order: $x_1, x_2, \dots$ Start with the procedure:

Step 1. Since $|f_n(x_1)| \leq D\ \forall\, n$, $\exists$ a subsequence $\{n_{l,1}\}$: $\{f_{n_{l,1}}(x_1)\}$ converges (denote the limit by $f(x_1)$).

...

Step $r+1$. Assume we have defined a sequence $\{n_{l,r}\}$: $f_{n_{l,r}}(x_i) \to f(x_i) \quad \forall\, i = 1, \dots, r$. Since $|f_{n_{l,r}}(x_{r+1})| \le D \;\forall\, l$, $\Longrightarrow \;\; \exists$ a subsequence $\{n_{l,r+1}\}$: $\{f_{n_{l,r+1}}(x_{r+1})\}$ converges (denote the limit by $f(x_{r+1})$).

...

Thus, $\forall\; r = 1, 2, \dots$, we defined a sequence $\{n_{l,r}\}$: $f_{n_{l,r}}(x_i) \to f(x_i) \quad \forall\; i = 1, \dots, r$.

Now: set $n_r = n_{r,r}$.

Note: $\forall\, r_0$, $\{n_r, r \ge r_0\}$ is a subsequence of $\{n_{l,r_0}\}$

$$\Longrightarrow \quad f_{n_r}(x_{r_0}) \to f(x_{r_0}).$$

□

Corollary 1.

(I). *Assume, in addition, that* $\exists\, C < \infty$: $\forall\, x, y \in [-d, d]$,

$$\limsup_{n\to\infty} |f_n(x) - f_n(y)| \le C|x - y|.$$

If f is any limit from Lemma 3, then

$$|f(x) - f(y)| \le C|x - y|, \quad \forall \;\text{ rational } x, y \in [-d, d].$$

Therefore,

(i) $\forall\; t \in [-d, d]$, *one can define* $f(t) = \lim_{x_l \to t} f(x_l)$, *where* $\{x_l\}$ *are rational;*

(ii) *this limit does not depend on* $\{x_l\}$;

(iii) $f(t)$ *is continuous in* $[-d, d]$ *and*

$$|f(t_1) - f(t_2)| \le C|t_1 - t_2|.$$

(II). *If conditions (I) are satisfied* $\forall\; d$ *and if all* $\{f_n\}$ *are non-decreasing, then*

$$f_{n_r} \overset{\text{u.o.c.}}{\to} f.$$

Problem No 1. Prove the corollary.

Definition 2. *A function f is* *Lipshitz continuous with parameter $C < \infty$* *if* $\forall\, x, y$

$$|f(x) - f(y)| \le C|x - y|.$$

Theorem 1. *Let $f : \mathbf{R} \to \mathbf{R}$ (or $f : \mathbf{R}_+ \to \mathbf{R}$) be Lipshitz continuous. Then*

(a) $\exists$ *a measurable set* $B \equiv B(f) \subseteq \mathbf{R}$ *(or* $\subseteq \mathbf{R}_+$*)* : $\lambda(B) = 0$ *and*

$$\forall\, x \notin B, \quad \exists\, f'(x) = \lim_{\Delta \to 0} \frac{f(x+\Delta) - f(x)}{\Delta}, \quad |f'(x)| \le C.$$

(b) *Let* $f'(x) = 0$ *for* $x \in B$. *Then* $\forall\, x, \quad \forall\, t > 0$

$$f(x+t) - f(x) = \int_x^{x+t} f'(z)dz.$$

Definition 3. *Any point* $x \notin B$ *is a* <u>*regular point*</u> *of* f.

Corollary 2. *Let* $f : \mathbf{R}_+ \to \mathbf{R}_+$ *be Lipshitz continuous,* $f(0) > 0$.*Assume* $\exists\, \varepsilon > 0$:
$\forall$ *regular point* $t \ge 0$, *if* $f(t) > 0$, *then* $f'(t) \le -\varepsilon$.
Then $\exists\, t_0 \le \dfrac{f(0)}{\varepsilon}$: $f(t) = 0 \;\; \forall\, t \ge t_0$.

Proof.

First, if $f(t_0) = 0$, then $f(t) = 0 \quad \forall\, t \ge t_0$, since $f(t) - f(t_0) = \int_{t_0}^t f'(z)dz \le 0$.

Second, if $f(t) > 0 \quad \forall\, t \le \dfrac{f(0)}{\varepsilon}$, then

$$f\left(\frac{f(0)}{\varepsilon}\right) \le \int_0^{f(0)/\varepsilon} (-\varepsilon)dz + f(0) \le 0.$$

□