Lectures on Stochastic Stability

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Lecture 2

Lyapunov functions: Criteria for Positive Recurrence and for Instability

1 History

Originally, Lyapunov functions were developed by A. Lyapunov in 1899 for the study of stability of dynamical systems described by ODE's, mainly motivated by mechanical systems. Since then, the methods based on Lyapunov functions have been extended to study stability of dynamical systems of all kinds (chaotic systems, control systems, stochastic systems, discrete systems, etc.)

2 Introduction

Our goal in this part of the lectures is to exemplify why and how Lyapunov functions work for (Harris) Markov processes in a general state space. To avoid technicalities (which have not fully resolved) we shall avoid continuous time and follow a more-or-less common convention that a *Markov chain* is any discrete-time Markov process in a general state space.

The field of applications that interests us is those stochastic systems that appear to have "simple evolution" "away from boundaries". Highly-nonlinear but smooth stochastic systems are also interesting but more well-understood than our cases.

3 Lyapunov functions for Markov chaines

The problem of (stochastic) stability for a stochastic system described by a Markov chain often (most of the time?) boils down to proving that some set is positive recurrent.

The setup Our object of study is a Markov chain (X_n) with values in some general state space S which will be assumed to be Polish (i.e. complete separable metric space). A time-homogeneous Markov chain with values in S can be described either by its transition probability kernel

$$\mathbf{P}(x, B) = \mathbf{P}(X_{n+1} \in B \mid X_n = x) \tag{1}$$

or by a stochastic recursion

$$X_{n+1} = f(X_n, \xi_n), \tag{2}$$

where the ξ_n are i.i.d. random variables. The existence of this explicit representation is simple when S is countable (and technical when S is a Polish space). More precisely, such a representation (2) exists for any time-homogeneous Markov chain taking values in the state space whose sigma-algebra is *countably generated*.

Stationarity (**stationary regime, stationary solution**, etc.) This means existence of a stationary Markov chain with the given transition kernel or, equivalently, existence of a stationary probability measure π on S:

$$\pi(B) = \int_S \pi(dx) P(x, B).$$

Uniqueness is also often desirable.

Stochastic Stability This means *stabilisation in time*, or *convergence* of X_n, as $n \to \infty$, in some stochastic sense.

We first realise that a.s. convergence ("in forward time") is impossible, except in trivial situations when we have absorbing states.

The weakest notion of convergence is that, for some x, the probability $\mathbf{P}_x(X_n \in \cdot)$ converges (weakly), as $n \to \infty$, to some proper probability distribution.

A stonger notion is to claim the above for all $x \in S$.

An even stronger notion is convergence in total variation.

Drift Let $V : S \to \mathbb{R}_+$ be some function on the state space S of a Markov chain (X_n). The drift of V in n steps is defined by

$$DV(x, n) := \mathbf{E}_x[V(X_n) - V(X_0)] = \mathbf{E}[V(X_n) - V(X_0) \mid X_0 = x]$$

(provided the expectations exist) There is a space and a time argument in $DV(x, n)$. It is much more general and convenient to define the drift for a state-dependent time-horizon, i.e. make n a function of x.

So, given a function $g : S \to \mathbb{N}$, we let

$$DV(x, g) := \mathbf{E}_x[V(X_{g(x)}) - V(x)].$$

(Positive) Recurrence of a set For a measurable set $B \subseteq S$ define

$$\tau_B = \inf\{n \geq 1 : \; X_n \in B\}$$

to be the first *return time*[1] to B if $X_0 \in B$ and the first *hitting time*, otherwise.

- The set B is called *recurrent* if

$$\mathbf{P}_x(\tau_B < \infty) = 1, \quad \text{for all } x \in B.$$

- It is called *positive recurrent* if

$$\sup_{x \in B} \mathbf{E}_x \tau_B < \infty.$$

It is this last property that is determined by a suitably designed Lyapunov function.

Roughly speaking, we want to prove that if we can find a function V (the Lyapunov function) such that the drift is negative outside a set, then the set is positive recurrent.

This is the content of Theorems 1 and 2 below. The first theorem is a particular case of the second. That this property can be translated into a stability statement is the subject of a next lecture.

Theorem 1. *Suppose that the drift of V in one step satisfies, for some positive N_0, c, and H,*

$$\mathbf{E}_x[V(X_1) - V(X_0)] \leq -c \quad \text{if} \quad V(x) > N_0$$

and

$$\mathbf{E}_x[V(X_1) - V(X_0)] \leq H < \infty \quad \text{if} \quad V(x) \leq N_0$$

Then the set

$$B = \{x \; : \; V(x) \leq N_0\}$$

is positive recurrent.

Proof. We follow an idea that is due to Tweedie (1976). Let

$$\tau = \tau_B = \inf\{n \geq 1 \; : \; V(X_n) \leq N_0\} \leq \infty.$$

Let $\mathscr{F}_n$ be the sigma field generated by $X_0, \ldots, X_n$. Note that τ is a "predictable" stopping time in that $\mathbf{1}(\tau \geq i) \in \mathscr{F}_{i-1}$ for all i. We define the "cumulative energy" between 0 and $\tau \wedge n$ by

$$\mathscr{E}_n = \sum_{i=0}^{\tau \wedge n} V(X_i) = \sum_{i=0}^{n} V(X_i) \mathbf{1}(\tau \geq i),$$

[1] This τ_B is a random variable. Were we working in continuous time, this would not, in general, be true, unless the paths of X and the set B were sufficiently "nice" (another instance of what technical complexities may arise in a continuous-time setup).

and estimate the change $\mathbf{E}_x(\mathscr{E}_n - \mathscr{E}_0)$ (which is finite) in a "martingale fashion".[2] Assume that $X_0 \le N_0$. Then

$$\begin{aligned}
\mathbf{E}_x(\mathscr{E}_n - \mathscr{E}_0) &= \mathbf{E}_x \sum_{i=1}^{n} \mathbf{E}_x(V(X_i)\mathbf{1}(\tau \ge i) \mid \mathscr{F}_{i-1}) \\
&= \mathbf{E}_x \sum_{i=1}^{n} \mathbf{1}(\tau \ge i)\mathbf{E}_x(V(X_i) \mid \mathscr{F}_{i-1}) \\
&\le V(x) + H + \mathbf{E}_x \sum_{i=2}^{n} \mathbf{1}(\tau \ge i)(V(X_{i-1}) - c) \\
&\le V(x) + H + \mathbf{E}_x \sum_{i=2}^{n+1} \mathbf{1}(\tau \ge i-1)V(X_{i-1}) - c\mathbf{E}_x \sum_{i=2}^{n} \mathbf{1}(\tau \ge i) \\
&= V(x) + H + c + \mathbf{E}_x\mathscr{E}_n - c\mathbf{E}_x \sum_{i=1}^{n} \mathbf{1}(\tau \ge i) \\
&= V(x) + H + c + \mathbf{E}_x\mathscr{E}_n - c\mathbf{E}_x \min(n, \tau),
\end{aligned}$$

where we used that $\mathbf{E}_x[(V(X_1) - V(X_0) \mid \mathscr{F}_0] \le H$ and $\mathbf{E}_x[(V(X_i) - V(X_{i-1}) \mid \mathscr{F}_{i-1}] \le -c$ if $\tau \ge i \ge 1$. We also used $\mathbf{1}(\tau \ge i) \le \mathbf{1}(\tau \ge i-1)$ and replaced n by $n+1$ in the pre-last inequality.

Since $\mathbf{E}_x\mathscr{E}_0 = \mathbf{E}_x V(x) = V(x)$, we obtain

$$\mathbf{E}_x \min(\tau, n) \le (V(x) + H + c)/c, \quad \text{for any} \quad n.$$

Letting n to infinity and using the motonote convergence theorem, we have

$$\mathbf{E}_x\tau \le (V(x) + H + c)/c \le (N_0 + H + c)/c. \tag{3}$$

□

Now we formulate and prove a statement which contains a general result on the positive recurrence of a certain set. For that, we impose a number of assumptions.

Assumptions

- **(L0)** V is unbounded from above: $\sup_{x\in S} V(x) = \infty$.
- **(L1)** h is bounded from below: $\inf_{x\in S} h(x) > -\infty$.
- **(L2)** h is eventually positive: $\underline{\lim}_{V(x)\to\infty} h(x) > 0$.
- **(L3)** g is locally bounded from above: $G(N) = \sup_{V(x)\le N} g(x) < \infty, \quad$ for all $N > 0$.
- **(L4)** g is eventually bounded by h: $\overline{\lim}_{V(x)\to\infty} g(x)/h(x) < \infty$.

[2] albeit we do not make use of explicit martingale theorems

Theorem 2. *Suppose that the drift of V in $g(x)$ steps satisfies*

$$\mathbf{E}_x[V(X_{g(x)}) - V(X_0)] \le -h(x),$$

where V, g, h satisfy (L0)–(L4). *Let*

$$\tau \equiv \tau_N = \inf\{n \ge 1 : \ V(X_n) \le N\}.$$

Then there exists $N_0 > 0$, such that for all $N \ge N_0$ and any $x \in S$, we have

$$\mathbf{E}_x \tau < \infty,$$
$$\sup_{V(x) \le N} \mathbf{E}_x \tau < \infty.$$

Proof. From the drift condition, we obviously have that $V(x) - h(x) \ge 0$ for all x. We choose N_0 such that $\inf_{V(x)>N_0} h(x) > 0$. Then, for, $N \ge N_0$, we set

$$d = \sup_{V(x)>N} g(x)/h(x), \quad -H = \inf_{x \in S} h(x), \quad c = \inf_{V(x)>N} h(x).$$

We define an increasing sequence t_n of stopping times recursively by

$$t_0 = 0, \quad t_n = t_{n-1} + g(X_{t_{n-1}}), \quad n \ge 1.$$

By the strong Markov property, the variables

$$Y_n = X_{t_n}$$

form a (possibly time-inhomogeneous) Markov chain with, as easily proved by induction on n, $\mathbf{E}_x V(Y_{n+1}) \le \mathbf{E}_x V(Y_n) + H$, and so $\mathbf{E}_x V(Y_n) < \infty$ for all n and x. Define the stopping time

$$\gamma = \inf\{n \ge 1 : \ V(Y_n) \le N\} \le \infty,$$

for which

$$\tau \le t_\gamma, \ \text{a.s.},$$

and so proving $\mathbf{E}_x t_\gamma < \infty$ is enough. Let $\mathscr{F}_n$ be the sigma field generated by $Y_0, \ldots, Y_n$. Note that γ is a "predictable" stopping time in that $\mathbf{1}(\gamma \ge i) \in \mathscr{F}_{i-1}$ for all i. We define the "cumulative energy" between 0 and $\gamma \wedge n$ by

$$\mathscr{E}_n = \sum_{i=0}^{\gamma \wedge n} V(Y_i) = \sum_{i=0}^{n} V(Y_i) \mathbf{1}(\gamma \ge i),$$

and estimate again the change $\mathbf{E}_x(\mathscr{E}_n - \mathscr{E}_0)$:

$$\begin{aligned}
\mathbf{E}_x(\mathscr{E}_n - \mathscr{E}_0) &= \mathbf{E}_x \sum_{i=1}^{n} \mathbf{E}_x(V(Y_i)\mathbf{1}(\gamma \geq i) \mid \mathscr{F}_{i-1}) \\
&= \mathbf{E}_x \sum_{i=1}^{n} \mathbf{1}(\gamma \geq i)\mathbf{E}_x(V(Y_i) \mid \mathscr{F}_{i-1}) \\
&\leq \mathbf{E}_x \sum_{i=1}^{n} \mathbf{1}(\gamma \geq i)(V(Y_{i-1}) - h(Y_{i-1})) \\
&\leq \mathbf{E}_x \sum_{i=1}^{n+1} \mathbf{1}(\gamma \geq i-1)V(Y_{i-1}) - \mathbf{E}_x \sim_{i=1}^{n} h(Y_{i-1}) \\
&= \mathbf{E}_x\mathscr{E}_n - \mathbf{E}_x \sum_{i=0}^{n-1} h(Y_i)\mathbf{1}(\gamma \geq i),
\end{aligned}$$

where we used that $V(x) - h(x) \geq 0$ and, for the pre-last inequality, we also used $\mathbf{1}(\gamma \geq i) \leq \mathbf{1}(\gamma \geq i-1)$ and replaced n by $n+1$. From this we obtain

$$\mathbf{E}_x \sum_{i=0}^{n-1} h(Y_i)\mathbf{1}(\gamma \geq i) \leq \mathbf{E}_x V(X_0) = V(x). \tag{4}$$

Assume first $V(x) > N$. Then $V(Y_i) > N$ for $i < \gamma$, by the definition of γ, and so

$$h(Y_i) \geq c > 0, \quad \text{for } i < \gamma, \tag{5}$$

by the definition of c. Use (5) in (4) to obtain

$$c\mathbf{E}_x \sum_{i=0}^{n} \mathbf{1}(\gamma > i) \leq V(x) + H + c.$$

Using the monotone convergence theorem and (5), we have

$$c\mathbf{E}_x\gamma \leq V(x) + H + c < \infty.$$

Using $h(x) \geq dg(x)$ for $V(x) > N$, we also have

$$\sum_{i=0}^{\gamma-1} h(Y_i) \geq d \sum_{i=0}^{\gamma-1} g(Y_i) = dt_\gamma,$$

whence $t_\gamma < \infty$, a.s., and so

$$\mathbf{E}_x\tau \leq \mathbf{E}_x t_\gamma \leq \frac{V(x) + H + c}{cd}.$$

It remains to see what happens if $V(x) \leq N$. By conditioning on Y_1, we have

$$\mathbf{E}_x\tau \leq g(x) + \mathbf{E}_x((cd)^{-1}(V(Y_1) + c)\mathbf{1}(V(Y_1) > N)) \leq G(N) + \frac{N + G(N)H + c}{cd}.$$

□

Discussion: The theorem we just proved shows that the set $B_N = \{x \in S : \; V(x) \le N\}$ is *positive recurrent*. It is worth seeing that the theorem is a generalization of many more standard methods.

I. Pakes's lemma: This is the case above with $S = \mathbb{Z}$, $g(x) = 1$ and $h(x) = \varepsilon - C_1 \mathbf{1}(V(x) \le C_2)$.

II. The Foster-Lyapunov criterion: Here S is general, and $g(x) = 1$ and $h(x) = c - H\mathbf{1}(V(x) \le N_0)$,

III. Dai's criterion: When $g(x) = \lceil V(x) \rceil$ (where $\lceil t \rceil = \inf\{n \in \mathbb{N} : \; t \le n\}$, $t > 0$), and $h(x) = \varepsilon V(x) - C_1 \mathbf{1}(V(x) \le C_2)$, we have Dai's criterion which is the same as the "fluid limits" criterion. More on this will be seen later.

IV. The Meyn-Tweedie criterion: When $h(x) = g(x) - C_1 \mathbf{1}(V(x) \le C_2)$ we have the Meyn-Tweedie criterion.

V. Fayolle-Malyshev-Menshikov: Similar state-dependent drift conditions, for countable Markov chains, were considered by these authors.

The indispensability of the "technical" conditions. It is clear why (L0)–(L3) are needed. As for condition (L4), this is not only a technical condition. Its indispensability can be seen in the following simple example: Consider $S = \mathbb{N}$, and transition probabilities

$$p_{1,1} = 1, \quad p_{k,k+1} \equiv p_k, \quad p_{k,1} = 1 - p_k \equiv q_k, \quad k = 2, 3, \ldots,$$

where $0 < p_k < 1$ for all $k \ge 2$ and $p_k \to 1$, as $k \to \infty$. Thus, jumps are either of size $+1$ or $-k$, till the first time state 1 is hit. Assume

$$q_k = 1/k, \quad k \ge 2.$$

Then 1 is an absorbing state, and there is $C > 0$, such that

$$P(X_{n+1} = X_n + 1 \text{ for all } n) \le C \exp\left(-\sum_k q_k\right) = 0.$$

But, for $\tau = \inf\{n : \; X_n = 1\}$,

$$\sum_n P(\tau \ge n) \ge \sum_n \exp \sum_2^n q_i \sim \sum_n \frac{1}{n} = \infty.$$

Therefore, the Markov chain cannot be positive recurrent. Take now

$$V(k) = \log(1 \vee \log k), \quad g(k) = k^2.$$

We can estimate the drift and find

$$E_k[V(X_{g(k)}) - V(k)] \le -h(k), \tag{6}$$

where $h(k) = c_1 V(k) - c_2$, and c_1, c_2 are positive constants. It is easily seen that (L0)-(L3) hold, but (L4) fails. This makes Theorem 2 inapplicable in spite of the negative drift (6). Physically, the time horizon $g(k)$ over which the drift was computed is far too large compared to the estimate $h(k)$ for the size of the drift itself.

Now we discuss some **examples.**

4 Instability criteria

We consider here two concepts of *instability*: the members of a certain class of sets either (i) cannot be positive recurrent or (b) are transient (the definition of *transience* will be given).

We first give a simple instability criterion due to Tweedie which gives conditions for a Markov chain not to be positive recurrent.

Theorem 3. *Suppose there is a non-constant function $V : S \to \mathbb{R}_+$ such that*
(a) $\sup x \in SDV(x, 1) < \infty$ and
(b) $DV(x, 1) \geq 0$ when $V(x) \geq K$, for some $K > 0$. Then the Markov chain cannot be positive recurrent.

Proof. The first condition implies that $\mathbf{E}_x|V(X_n)| < \infty$ for all $x \in S$. Now let $\tau = \inf\{n : V(X_n) < K\}$. The second condition can be written as

$$(V(X_{\tau\wedge n}), n \geq 0) \text{ is a submartingale under } \mathbf{P}_x, \quad \text{for all} \quad x \geq K.$$

Hence

$$\mathbf{E}_x V(X_{\tau\wedge n}) \geq \mathbf{E}_x V(X_{\tau\wedge 0}) = V(x) \geq K.$$

If the Markov chain is positive recurrent then $\mathbf{E}_x\tau < \infty$ and, by the martingale convergence theorem, $V(X_{\tau\wedge n}) \to V(X_\tau)$ in L_1. Therefore,

$$\mathbf{E}_x V(X_\tau) \geq K.$$

But $V(X_\tau) < K$ a.s., and so we arrived at a contradiction. □

We now pass to transience.

Transient set A set $B \subseteq S$ is called *transient* if $\mathbf{P}_x(\tau_B = \infty) > 0$ for all $x \in S$, where $\tau_B = \inf\{n \geq 1 : \ X_n \in B\}$ is the first return time to B.

Let $V : S \to \mathbb{R}_+$ be a "norm-like" function, i.e., suppose (at least) that V is unbounded. We say that the chain is transient if each set of the form $B_N = \{x \in S : \ V(x) \leq N\}$ is transient.

We now introduce a general criterion that decides whether $\lim_{n\to\infty} V(X_n) = \infty$, $\mathbf{P}_x$-a.s. Clearly then, this will imply transience of each B_N.

Thinking of V as a Lyapunov function, it is natural to seek criteria that are, in a sense, opposite to those of Theorem 2. One would expect that if the drift $\mathbf{E}_x[V(X_1) - V(X_0)]$ is bounded from below by a positive constant, outside a set of the form B_N, then that would imply instability. However, this is not true and this has been a source of difficulty in formulating a general enough criterion thus far. To the best of our knowledge, the most general criterion which may be found in textbooks is Theorem 2.2.7. of Fayolle et al. (1995) which is, however, rather restrictive because (i) it is formulated for countable state Markov chains and (ii) it requires that a transition from a state x to a state y, with $V(x) - V(y)$ larger than a certain constant, is not possible. However, it gives insight as to what problems one might encounter: one needs to regulate, not only the drift from below, but also its size when the drift is large.

The theorem below is a generalization of the one mentioned above. First, define

$$\sigma_N := \tau_{B_N^c} = \inf\{n \geq 1 : \ V(X_n) > N\}$$
$$\Delta_x := V(X_1) - V(X_0), \quad \text{given} \quad X_0 = x.$$

We then have:

Theorem 4. *Suppose there exist $N, M, \varepsilon > 0$ and a measurable $h : [0,\infty) \to \ [1,\infty)$ with the property that $h(t)/t$ be increasing on $1 \leq t < \infty$, and $\int_1^\infty h(t)^{-1} dt < \infty$, such that*

- **(I1)** $\mathbf{P}_x(\sigma_N < \infty) = 1$ *for all x.*
- **(I2)** $\inf_{x \in B_N^c} \mathbf{E}_x[\Delta, \Delta \leq M] \geq \varepsilon$.
- **(I3)** *The family $\{h^-(\Delta), x \in B_N^c\}$ is uniformly integrable.*

Then $\mathbf{P}_x(\lim_{n\to\infty} V(X_n) = \infty) = 1$, for all $x \in S$.

Possible examples of function h are $x^{1+\alpha}$, with positive α, and $x \ln^2 x$.

This theorem is proved in F–Denisov (2001) under a minor extra technical condition which may be omitted . The proof is based on the super-martingale convergence theorem. First, for an appropriate function g (such that $g(x) \to 0$ implies that $x \to \infty$),
(a) we show that r.v.s $\{g(V_n), n = 1, 2, \ldots\}$ form a positive super-martingale and, therefore, converge a.s. to an a.s. finite limit,
(b) we find a (random) subsequence $n_1 < n_2 < \ldots$ such that $g(V_{n_k})$ converges to zero a.s. Then the result follows.

We remark that there are extensions for non-homogeneous Markov chains. Condition (I1) says that the set B_N^c is recurrent. Of course, if the chain itself forms one communicating class, then this condition is automatic. Condition (I2) is the positive drift condition. Condition (I3) is the condition that regulates the size of the drift. We also note that an analogue of this theorem, with state-dependent drift can also be derived. (The theorem of Fayolle et al. does use state-dependent drift.)

To see that (I3) is essential, consider the following example: Let $S := \mathbb{Z}_+$, and $\{X_n\}$ a Markov chain with transition probabilities

$$p_{i,i+1} = 1 - p_{i,0}, \quad i \geq 1,$$
$$p_{0,1} = p_{0,0} = 1/2.$$

Suppose that $0 < p_{i,0} < 1$ for all i, and $\sum_i p_{i,0} < \infty$. Then the chain forms a single communicating class. Also, with τ_0 the first return to 0, we have

$$P_i(\tau_0 = \infty) = \prod_{j \geq i} (1 - p_{j,0}) > 0.$$

So the chain is transient. However not that the natural choice for V, namely $V(x) \equiv x$ trivially makes (I2) true.

Other **examples**:...

5 Topics for discussion and problems

1. Finding an appropriate Lyapunov function is an art. Discuss how the geometric picture may help you guess the form of a Lyapunov function.

2. Consider (ξ_n) to be i.i.d. Gaussian, say, random vectors in $\mathbb{R}^d$ with zero mean, and define the Markov chain

$$X_{n+1} = AX_n + \xi_n,$$

   where A is a $d \times d$ matrix with eigenvalues having magnitude strictly smaller than 1. Show that the unit ball is positive recurrent by means of an appropriate Lyapunov function.

3. Now let $d = 1$ and, instead of A, consider a time-dependent A_n, where (A_n) are i.i.d. positive random variables:

$$X_{n+1} = A_n X_n + \xi_n.$$

   Show that the unit ball is positive recurrent if $E \log A_1 < 0$.

4. (Lamperti criterion) Consider a Markov chain in $\mathbb{Z}_+$ with $E(X_{n+1} - X_n \mid X_n = x) \sim -c/x$, and $E((X_{n+1} - X_n)^2 \mid X_n) \to b$, as $x \to \infty$, where c, b are positive constants. Use an appropriate Lyapunov function in order to deduce that the chain is positive recurrent if $2c > b$. (Also prove that it is transient if $2c < b$. The "critical case" $2c = b$ is a tough one and what happens there depends on other conditions as well.)

5. The classical Lindley recursion is

$$X_{n+1} = (X_n + \xi_n)^+.$$

   Assume that the (ξ_n) are i.i.d. integrable random variables with $E\xi_1 < 0$. Show that the set $[0, b]$ is positive recurrent, for any $b \geq 0$.

6. Suppose X_t is a Markov Jump Process (i.e. a Markov process in continuous time with at most finitely many jumps on each bounded time interval, almost surely). Recall that such a process is defined (in distribution) through its transition rates $q(x, y)$, $x \neq y$. Formulate a Lyapunov function criterion directly in terms of the rates. (*Hint*: use any natural time embedding !)

7. A 2-station Jackson network may be represented as a continuous-time Markov process in $\mathbb{Z}_+^2$ with $q(x, x + e_1) = \lambda$, $q(x + e_1, x + e_2) = \mu_1$, $q(x + e_2, x) = \mu_2$, $x \in \mathbb{Z}_+^2$. Here $e_1 = (1, 0)$, $e_2 = (0, 1)$ are the standard unit vectors. Find Lyapunov function when $\lambda < \mu_1 < \mu_2$ that shows that the unit ball is positive recurrent. Repeat when $\lambda < \mu_2 < \mu_1$. (*Hint*: You may choose an appropriate time embedding, and linear or piecewise linear text functions. To do so, it is helpful to consider the geometric point of view).

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