273018 Special Functions, Homework #9, 16.3.2011

9.2.3 Given that

$$P_1(x) = x \quad \text{and} \quad Q_0(x) = \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

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are solutions of Legendre's differential equation corresponding to *different* eigenvalues:

(a) Evaluate their orthogonality integral

$$\int_{-1}^{1} \frac{x}{2} \ln\left(\frac{1+x}{1-x}\right) dx.$$

(b) Explain why these two functions are *not* orthogonal, why the proof of orthogonality does not apply.

9.2.5 (a) Show that the first derivatives of the Legendre polynomials satisfy a self-adjoint differential equation with eigenvalue $\lambda = n(n+1) - 2$.

(b) Show that these Legendre polynomial derivatives satisfy an orthogonality relation

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$$\int_{-1}^{1} P_m'(x)P_n'(x)(1-x^2)\,dx = 0, \qquad m \neq n.$$

Note. In Section 12.5, $(1-x^2)^{1/2}P_n'(x)$ will be labeled an associated Legendre polynomial, $P_n^1(x)$.

9.2.9 The ultraspherical polynomials $C_n^{(\alpha)}(x)$ are solutions of the differential equation

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$$\left\{(1-x^2)\frac{d^2}{dx^2} - (2\alpha+1)x\frac{d}{dx} + n(n+2\alpha)\right\}C_n^{(\alpha)}(x) = 0.$$

(a) Transform this differential equation into self-adjoint form.

(b) Show that the $C_n^{(\alpha)}(x)$ are orthogonal for different n. Specify the interval of integration and the weighting factor.

Note. Assume that your solutions are polynomials.

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$$f'' + f' + \lambda f = 0, \qquad f(0) = 0 = f(1).$$

With respect to which inner product are these eigenfunctions orthogonal (and over which interval)?

Warning: This equation is not self-adjoint.

```
function [ output_args ] = uppg  ( N )

x = -1:0.01:1;

c(N) = 0;
sum(length(x)) = 0;

% Calculates the coefficient for the Fourier-Legendre series
for i = 1:N
    c(i) = legendrecoeff(i);
end

% Calculates the value of the series up to order N
for j = 1:length(x)
    for i = 1:N
        leg = legendre(i,x(j));
        sum(j) = sum(j) + c(i)*leg(1);
    end
end

plot(x,sum);
```

```
% Returns the Fourier-Legendre coefficient of order n
function c = legendrecoeff(n)

    if (n==0)
        c = 0;
    else
        c = legendre2(n-1,0) - legendre2(n+1,0)
            + 0.5*(- legendre2(n-1,-1) + legendre2(n+1,-1) + legendre2(n+1,1) - legendr
e2(n-1,1));
    end

end

% Extracts the first element from Matlabs legendre-function, which returs
% the associated Legendre function
function leg = legendre2(r,x)

    leg = legendre(r,x);
    leg = leg(1);

end
```