273018 Special Functions, Homework # 8, 9.3.2011

12.4.13 In numerical work (such as the Gauss–Legendre quadrature of Appendix 2) it is useful to establish that $P_n(x)$ has n real zeros in the interior of $[-1, 1]$. Show that this is so.

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Hint. Rolle's theorem shows that the first derivative of $(x^2 - 1)^{2n}$ has one zero in the interior of $[-1, 1]$. Extend this argument to the second, third, and ultimately to the nth derivative.

9.1.6 For the very special case $\lambda = 0$ and $q(x) = 0$ the self-adjoint eigenvalue equation becomes

$$\frac{d}{dx}\left[p(x)\frac{du(x)}{dx}\right] = 0,$$

satisfied by

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$$\frac{du}{dx} = \frac{1}{p(x)}.$$

Use this to obtain a "second" solution of the following:

(a) Legendre's equation,
(b) Laguerre's equation,
(e) Hermite's equation.

ANS (a) $u_2(x) = \frac{1}{2}\ln\frac{1+x}{1-x}$,

(b) $u_2(x) - u_2(x_0) = \int_{x_0}^{x} e^t \frac{dt}{t}$,

(c) $u_2(x) = \int_0^x e^{t^2}\,dt$.

These second solutions illustrate the divergent behavior usually found in a second solution.

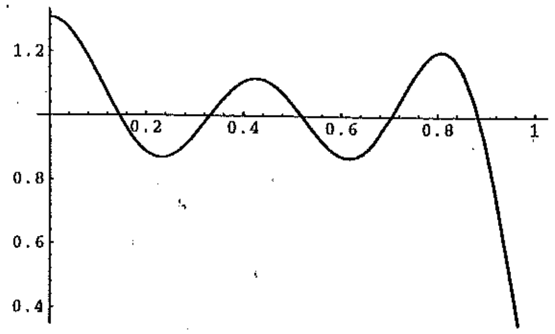
Note. In all three cases $u_1(x) = 1$.

~~9.1.7~~ 33 Given that $\mathfrak{L}u = 0$ and $g\mathfrak{L}u$ is self-adjoint, show that for the adjoint operator $\bar{\mathfrak{L}}$, $\bar{\mathfrak{L}}(gu) = 0$.

25 solution

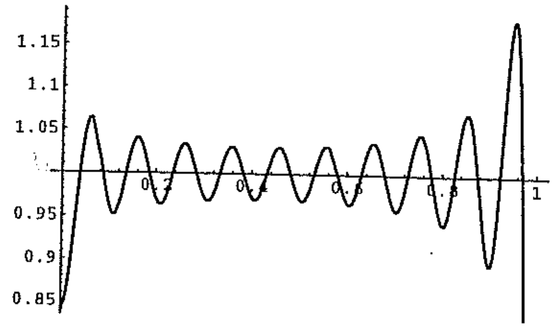
```
lambda[1] = FindRoot[BesselJ[0,x],
    {x, 2.4}][[1,2]]
2.40483

Do[lambda[k+1] = FindRoot[BesselJ[0,x],
    {x, lambda[k] + Pi}][[1,2]], {k, 49}]

Do[c[k] = 2 / (lambda[k] BesselJ[1, lambda[k]]), {k, 50}]

u[r_, t_, N_] := Sum[c[k] BesselJ[0, lambda[k] r] *
Exp[ -lambda[k]^2 t], {k, 1, N}]

f[r_, N_] := Sum[c[k] BesselJ[0, lambda[k] r], {k, 1, N}]
```

```
Plot[f[r, 5], {r, 0, 1}]
```



-Graphics-

```
Plot[f[r, 20], {r, 0, 1}]
```



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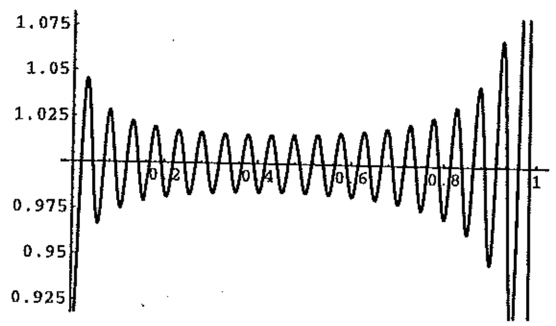
```
Plot[f[r, 40], {r, 0, 1}]
```



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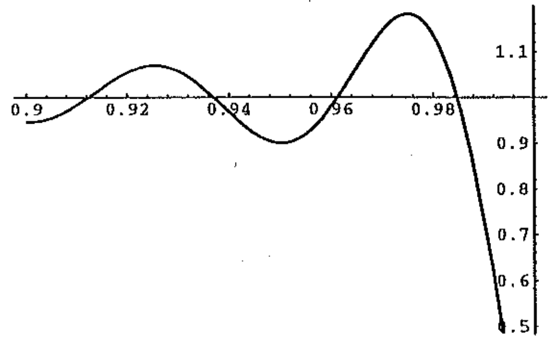
```
Plot[f[r, 40], {r, 0.9, 1}]
```



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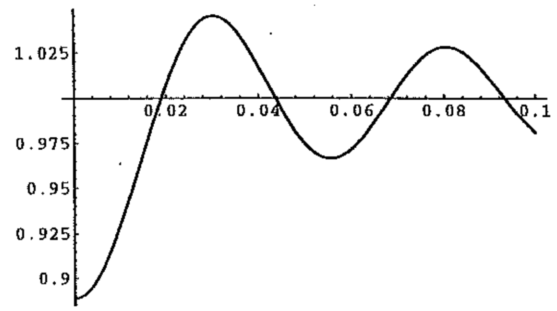
```
Plot[f[r, 40], {r, 0, 0.1}]
```



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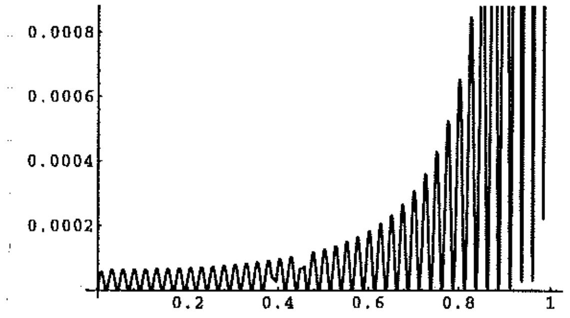
```
Plot[r (f[r, 40] - 1)^2, {r, 0, 1}]
```



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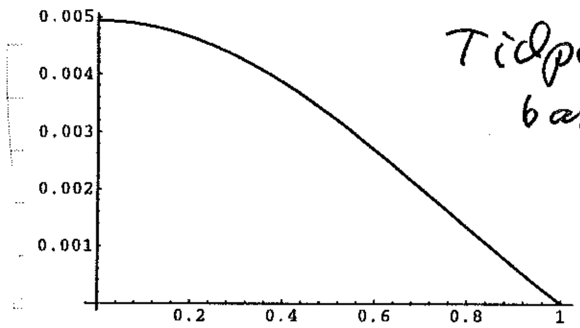
```
u[r_, t_, N_] := Sum[c[k] BesselJ[0, lambda[k] r] *
Exp[ -lambda[k]^2 t], {k, 1, N}]

Plot[u[r, 1, 1], {r, 0, 1}]
```



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bara en
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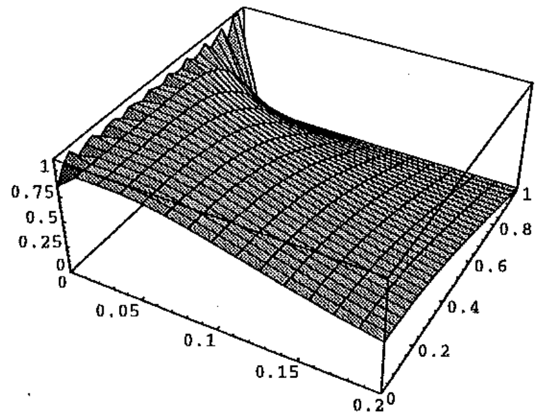
```
g0 = Plot3D[u[r, t, 20], {t, 0, 0.2}, {r, 0, 1},
    PlotPoints -> {15, 30}]
```



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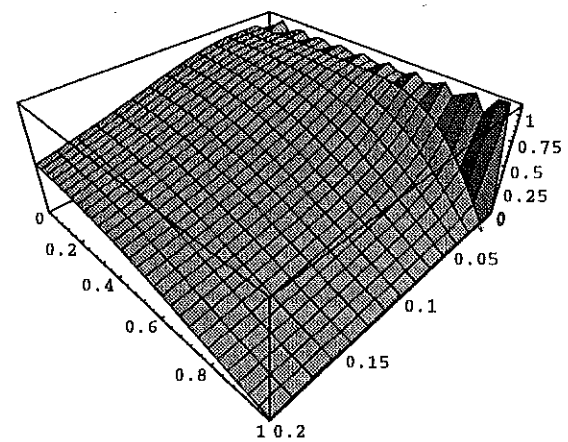
```
g0 = Show[g0, ViewPoint -> {1, 1, 1}]
```



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