Special Functions, Homework #6, Due 23.2. 2011

9.3.3 Apply the Gram–Schmidt procedure to form the first three Laguerre polynomials

$$u_n(x) = x^n, \qquad n = 0, 1, 2, \ldots,$$
$$0 \le x < \infty,$$
$$w(x) = e^{-x}.$$

(23) The conventional normalization is

$$\int_0^\infty L_m(x)L_n(x)e^{-x}\,dx = \delta_{mn}.$$

ANS. $L_0 = 1$,
$L_1 = 1 - x$,
$L_2 = \dfrac{(2 - 4x + x^2)}{2}$.

9.3.5 Using the Gram–Schmidt orthogonalization procedure, construct the lowest three Hermite polynomials:

$$u_n(x) = x^n, \qquad n = 0, 1, 2, \ldots, \qquad -\infty < x < \infty, \quad w(x) = e^{-x^2}.$$

(24) For this set of polynomials the usual normalization is

$$\int_{-\infty}^\infty H_m(x)H_n(x)w(x)\,dx = \delta_{mn}2^m m!\pi^{1/2}.$$

ANS. $H_0 = 1$,
$H_1 = 2x$,
$H_2 = 4x^2 - 2$.

(25) In the lecture notes we solved the heat equation with initial value $u(x) \equiv 1$ in a cylinder with radius $R=1$ and zero boundary condition to get (formally):

$$u(\rho, t) = \sum_{k=1}^\infty c_k J_0(\alpha_k \rho)\, e^{-\alpha_k^2 t}$$

$$c_k = \frac{2}{\alpha_k J_1(\alpha_k)}, \qquad \alpha_k = \text{zeros of } J_0.$$

a) Check numerically how well this series satisfies the initial condition $u(\rho, 0) \equiv 1$ (compute a reasonable number of terms). What happens close to the surface ($\rho = 1$).

b) Substitute $t=1$ to get an approximation of the solution at time $t=1$.

(26) The sequence of functions $J_0(\alpha_k \rho)$ in (25) is orthogonal and complete (but not normalized). Use this fact and Parseval's identity to compute $\sum_{k=1}^\infty 1/\alpha_k^2$

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In[3]:= Plot[BesselJ[0, x], {x, 0, 20}]
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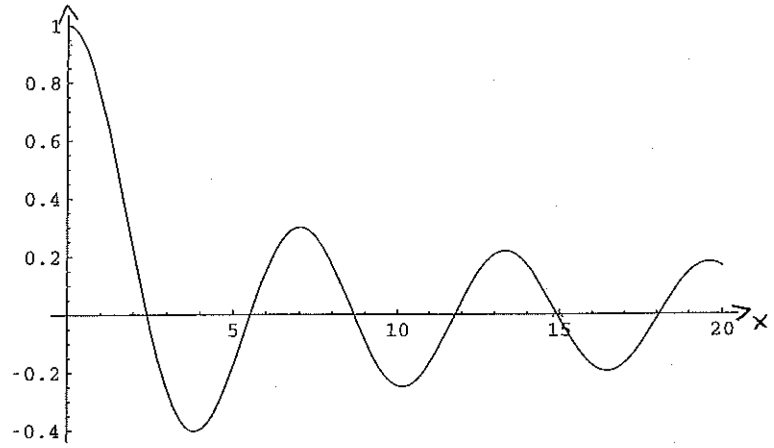


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Out[3]= - Graphics -

In[19]:= FindRoot[BesselJ[0, x] == 0, {x, 1}]
Out[19]= {x → 2.40483}

In[20]:= FindRoot[BesselJ[0, x] == 0, {x, 5}]
Out[20]= {x → 5.52008}

In[21]:= FindRoot[BesselJ[0, x] == 0, {x, 9}]
Out[21]= {x → 8.65373}

In[22]:= FindRoot[BesselJ[0, x] == 0, {x, 12}]
Out[22]= {x → 11.7915}

In[23]:= FindRoot[BesselJ[0, x] == 0, {x, 15}]
Out[23]= {x → 14.9309}

In[24]:= FindRoot[BesselJ[0, x] == 0, {x, 18}]
Out[24]= {x → 18.0711}

In[27]:= 5.52008 - 2.40483
Out[27]= 3.11525

In[28]:= 8.65373 - 5.52008
Out[28]= 3.13365

In[29]:= 11.7915 - 8.65373
Out[29]= 3.13777

In[30]:= 14.9309 - 11.7915
Out[30]= 3.1394

In[32]:= 18.0711 - 14.9309
Out[32]= 3.1402
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