Special Functions, Home Work #5, due 16.2.2011

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11.7.10 Derive Eq. (11.174):

$$\int_{-\infty}^{\infty} j_m(x) j_n(x)\, dx = 0, \qquad \begin{matrix} m \neq n \\ m, n \geq 0. \end{matrix}$$

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11.7.11 Derive Eq. (11.175):

$$\int_{-\infty}^{\infty} [j_n(x)]^2\, dx = \frac{\pi}{2n+1}.$$

Above j_n is the spherical Bessel function $j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x)$

14.3.1 Develop the Fourier series representation of

21 a)

$$f(t) = \begin{cases} 0, & -\pi \leq \omega t \leq 0, \\ \sin \omega t, & 0 \leq \omega t \leq \pi. \end{cases}$$

This is the output of a simple half-wave rectifier. It is also an approximation of the solar thermal effect that produces "tides" in the atmosphere.

$$ANS. \quad f(t) = \frac{1}{\pi} + \frac{1}{2}\sin \omega t - \frac{2}{\pi} \sum_{\substack{n=2,4,6,\ldots \\ \text{even}}}^{\infty} \frac{\cos n\omega t}{n^2 - 1}.$$

b) Plot a number of terms in the Fourier series. Does it look like the sum should converge uniformly (i.e., the maximal error tends to zero)?

14.3.2 A sawtooth wave is given by

22 a)

$$f(x) = x, \qquad -\pi < x < \pi.$$

Show that

$$f(x) = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

b) Same question as above in 21.

polar coordinates, the divergence was found to be $\operatorname{div}(F) = f_r + \frac{1}{r} f + \frac{1}{r} g_\theta$.

All these results are predicated on the *unit* basis vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$. Using this convention for basis vectors, Table 22.2 lists expressions for div, grad, curl, and laplacian in Cartesian and cylindrical coordinates, where $\mathbf{e}_z = \mathbf{k}$. Expressions for these operators in polar coordinates can be deduced from those in cylindrical coordinates. In Table 22.3, these same operators are listed for the two different versions of spherical coordinates found in the literature.

System	Cartesian	Cylindrical
Basis	$\mathbf{i}, \mathbf{j}, \mathbf{k}$	$\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z = \mathbf{k}$
Coordinates	x y z	$x = r\cos(\theta)$ $y = r\sin(\theta)$ $z = z$
grad ∇u	$\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}$	$\begin{bmatrix} u_r \\ \frac{1}{r} u_\theta \\ u_z \end{bmatrix}$
div $\nabla \cdot \mathbf{F}$	$f_x + g_y + h_z$	$\frac{1}{r}(rf)_r + \frac{1}{r} g_\theta + h_z$
curl $\nabla \times \mathbf{F}$	$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \begin{bmatrix} h_y - g_z \\ f_z - h_x \\ g_x - f_y \end{bmatrix}$	$\begin{vmatrix} \frac{\mathbf{e}_r}{r} & \mathbf{e}_\theta & \frac{\mathbf{e}_z}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ f & rg & h \end{vmatrix} = \begin{bmatrix} \frac{h_\theta - (rg)_z}{r} \\ f_z - h_r \\ \frac{(rg)_r - f_\theta}{r} \end{bmatrix}$
laplacian $\nabla^2 u$	$u_{xx} + u_{yy} + u_{zz}$	$\frac{(ru_r)_r}{r} + \frac{u_{\theta\theta}}{r^2} + u_{zz}$

TABLE 22.2 Vector operators in Cartesian and cylindrical coordinates

Symbols for the *unit* basis vectors in each system are provided *in the order in which the basis vectors are used.* Hence, in each coordinate system, vectors are of the form $\mathbf{F} = f\mathbf{e}_1 + g\mathbf{e}_2 + h\mathbf{e}_3$, where the ordering 1, 2, 3 is determined by the way the basis vectors are listed in the tables. For each system the gradient vector is given as a column vector. It is therefore essential that the order of the basis vectors be known. For each system the curl