273018 Special Function
Home Work #4, due 9.2.2011

(15) **11.1.1** From the product of the generating functions $g(x, t) \cdot g(x, -t)$ show that

$$1 = [J_0(x)]^2 + 2[J_1(x)]^2 + 2[J_2(x)]^2 + \cdots$$

and therefore that $|J_0(x)| \le 1$ and $|J_n(x)| \le 1/\sqrt{2}$, $n = 1, 2, 3, \ldots$.
Hint. Use uniqueness of power series, Section 5.7.

(16) **11.1.10** Derive

$$J_n(x) = (-1)^n x^n \left(\frac{d}{x\,dx}\right)^n J_0(x).$$

Hint. Try mathematical induction.

(17) a) Show that

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x, \quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

b) Use a suitable recurrence equation to compute $J_{3/2}(x)$ and $J_{5/2}(x)$.

(18) **11.1.12** An analysis of antenna radiation patterns for a system with a circular aperture involves the equation

$$g(u) = \int_0^1 f(r) J_0(ur) r\,dr.$$

If $f(r) = 1 - r^2$, show that

$$g(u) = \frac{2}{u^2} J_2(u).$$

SPHERICAL COORDINATES: $u = u(\rho, \phi, \theta)$, $\mathbf{v} = v_\rho \hat{\mathbf{e}}_\rho + v_\phi \hat{\mathbf{e}}_\phi + v_\theta \hat{\mathbf{e}}_\theta$

$$\nabla u = \left(\hat{\mathbf{e}}_\rho \frac{\partial}{\partial \rho} + \hat{\mathbf{e}}_\phi \frac{1}{\rho}\frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_\theta \frac{1}{\rho \sin\phi}\frac{\partial}{\partial\theta}\right) u = \frac{\partial u}{\partial \rho}\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial u}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{1}{\rho\sin\phi}\frac{\partial u}{\partial\theta}\hat{\mathbf{e}}_\theta$$

$$\nabla^2 u = \frac{1}{\rho^2}\left[\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right]$$

$$\nabla\cdot\mathbf{v} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}(\rho^2 v_\rho) + \frac{1}{\rho\sin\phi}\frac{\partial}{\partial\phi}(v_\phi\sin\phi) + \frac{1}{\rho\sin\phi}\frac{\partial v_\theta}{\partial\theta}$$

$$\nabla\times\mathbf{v} = \frac{1}{\rho\sin\phi}\left[\frac{\partial}{\partial\phi}(v_\theta\sin\phi) - \frac{\partial v_\phi}{\partial\theta}\right]\hat{\mathbf{e}}_\rho + \frac{1}{\rho}\left[\frac{1}{\sin\phi}\frac{\partial v_\rho}{\partial\theta} - \frac{\partial(\rho v_\theta)}{\partial\rho}\right]\hat{\mathbf{e}}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho v_\phi)}{\partial\rho} - \frac{\partial v_\rho}{\partial\phi}\right]\hat{\mathbf{e}}_\theta$$

$$= \frac{1}{\rho^2\sin\phi}\begin{vmatrix} \hat{\mathbf{e}}_\rho & \rho\hat{\mathbf{e}}_\phi & \rho\sin\phi\,\hat{\mathbf{e}}_\theta \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial\theta} \\ v_\rho & \rho v_\phi & \rho\sin\phi\, v_\theta \end{vmatrix}$$

$$\mathbf{R} = \rho\hat{\mathbf{e}}_\rho, \quad d\mathbf{R} = d\rho\,\hat{\mathbf{e}}_\rho + \rho\,d\phi\,\hat{\mathbf{e}}_\phi + \rho\sin\phi\,d\theta\,\hat{\mathbf{e}}_\theta$$

$$dA = \begin{cases} \rho^2|\sin\phi|\,d\phi\,d\theta & \text{(constant-}\rho\text{ surface)} \\ \rho|\sin\phi|\,d\rho\,d\theta & \text{(constant-}\phi\text{ surface)}, \\ \rho\,d\rho\,d\phi & \text{(constant-}\theta\text{ surface)} \end{cases} \quad dV = \rho^2\,|\sin\phi|\,d\rho\,d\phi\,d\theta$$

GAUSS DIVERGENCE THEOREM: $\int_{\mathcal{V}} \operatorname{div}\mathbf{v}\,dV = \int_{\mathcal{S}} \hat{\mathbf{n}}\cdot\mathbf{v}\,dA$

GREEN'S FIRST IDENTITY: $\int_{\mathcal{V}} (\nabla u\cdot\nabla v + u\nabla^2 v)\,dV = \int_{\mathcal{S}} u\frac{\partial v}{\partial n}\,dA$

GREEN'S SECOND IDENTITY: $\int_{\mathcal{V}} (u\nabla^2 v - v\nabla^2 u)\,dV = \int_{\mathcal{S}} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)dA$

STOKES' THEOREM: $\int_{\mathcal{S}} \hat{\mathbf{n}}\cdot\operatorname{curl}\mathbf{v}\,dA = \oint_{\mathcal{C}} \mathbf{v}\cdot d\mathbf{R}$

GREEN'S THEOREM: $\int_{\mathcal{S}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \oint_{\mathcal{C}} (P\,dx + Q\,dy)$