2.73018 Special Functions, Homework #10, 23.3.2011

(38) **13.1.6** Prove that

$$|H_n(x)| \le |H_n(ix)|. \quad (x \text{ is real})$$

13.1.8 (a) Expand x^{2r} in a series of even order Hermite polynomials.
(b) Expand x^{2r+1} in a series of odd order Hermite polynomials.

(39) *ANS.* (a) $x^{2r} = \dfrac{(2r)!}{2^{2r}} \displaystyle\sum_{n=0}^{r} \dfrac{H_{2n}(x)}{(2n)!(r-n)!}$

(b) $x^{2r+1} = \dfrac{(2r+1)!}{2^{2r+1}} \displaystyle\sum_{n=0}^{r} \dfrac{H_{2n+1}(x)}{(2n+1)!(r-n)!}, \qquad r = 0, 1, 2, \ldots.$

Hint. Use a Rodrigues representation of $H_{2n}(x)$ and integrate by parts.

(40) Show that the Hermite functions

$$h_n(x) = e^{-x^2/2} H_n(x)$$

are eigenfunctions of the Sturm-Liouville problem

$$y'' + (-x^2 + \lambda y) = 0,$$

with eigenvalue $\lambda = 2n+1$. Which ortogonality condition do we get from the general theory?

(41) Compute the Fourier transform of $\phi(x) = x^{10} e^{-x^2/2}$ by first expanding this function into a Fourier-Hermite series and then using the fact that the Hermite functions are eigenfunctions of the Fourier transform. Also draw a picture of the result.

Hint: Homework (39) is useful here.

2. HERMITEN POLYNOMIT:

2.1. $w''(z) - 2zw'(z) + 2nw(z) = 0$

2.2 $H_n(z) = (-1)^n . e^{z^2} . \dfrac{d^n}{dz^n}(e^{-z^2}) = \displaystyle\sum_{k=0}^{[\frac{n}{2}]} \dfrac{(-1)^k n!(2z)^{n-2k}}{k!(n-2k)!}$

2.3. $H_0(z) = 1$

$H_1(z) = 2z$

$H_2(z) = -2 + 4z^2$

$H_3(z) = -12z + 8z^3$

$H_4(z) = 12 - 48z^2 + 16z^4$

$H_5(z) = 120z - 160z^3 + 32z^5$

2.4. $H_n'(z) = 2nH_{n-1}(z) \qquad , n \in \mathbb{N}_0$

2.5. $zH_n'(z) = nH_{n-1}'(z) + nH_n(z) \qquad , n \in \mathbb{N}_0$

2.6. $zH_n(z) = \frac{1}{2} H_{n+1}(z) + nH_{n-1}(z) \qquad , n \in \mathbb{N}_0$

2.7. $e^{z^2-(z-\xi)^2} = \displaystyle\sum_{n=0}^{\infty} \dfrac{H_n(z)}{n!} \xi^n$

2.8. $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} \cdot H_m(x) H_n(x)\,dx = 2^n n! \sqrt{\pi}\, \delta_{mn}$

2.9. $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} \cdot x^m H_n(x)\,dx = \begin{cases} 0 & , m<n \\ n!\sqrt{\pi} & ; m=n \end{cases}$

Lähde: Sneddon, I.N.: Special Functions of Mathematical Physics and Chemistry

Huom: Edellä $[\frac{n}{2}] = \begin{cases} \frac{n}{2} & \text{, kun n on parillinen ja} \\ \frac{n-1}{2} & \text{, kun n on pariton} \end{cases}$

Sopimus: $H_{-1} = 0$