Markov Chains (273023), Exercise session 7, Tue 5 March 2013.

Exercise 7.1. Show that the simple random walk on $\mathbb{Z}$ is recurrent.

Exercise 7.2. Show that the simple random walk on $\mathbb{Z}$ is null recurrent. Use either the result in Exercise 4.6 of this course or the approach proposed in the Exercise 21.8 in Levin, Peres, Wilmer, p. 286.

Exercise 7.3 (Levin, Peres, Wilmer, Ex. 21.2, p. 285). Suppose that P is a transition probability matrix of an irreducible Markov chain (with possibly countably many states). Show that if $\pi P = \pi$ for a probability distribution π, then $\pi(x) > 0$ for every $x \in \Omega$.

Exercise 7.4. Let P be a symmetric transition probability matrix of an irreducible Markov chain with $\mathbb{Z}$ as the state space. Show that there are no stationary probability distributions. Conclude that the chain is either null recurrent or transient.

Exercise 7.5 (Levin, Peres, Wilmer, Ex. 21.5, p. 286). Let P be an irreducible and aperiodic transition matrix on Ω. Let $\tilde{P}$ be the matrix on $\Omega \times \Omega$ defined by

$$\tilde{P}((x,y),(z,w)) = P(x,z)P(y,w), \ (x,y) \in \Omega \times \Omega, \ (z,w) \in \Omega \times \Omega.$$

Show that $\tilde{P}$ is irreducible. Show by an example that it is necessary to assume that P is aperiodic.

Exercise 7.6. Let $\Omega = \{1, 2, \ldots, 10\}$. Let X_t be a (hidden) Markov chain with transition probability matrix

$$P(i,j) = \frac{c_i}{1 + |i - j|}$$

where c_i are constants. Suppose that the observation U_t of X_t follows the distribution

$$U(i,j) = \mathbb{P}(U_t = i | X_t = j) = \frac{d_i}{2 + |i - j|}$$

where d_i are suitable constants. What is the probability that if we observe the sequence

$$(U_t)_{t=1}^{10} = (2, 4, 6, 8, 10, 2, 4, 6, 8, 10)$$

the Markov chain X_t really passed through these states, i.e.

$$(X_t)_{t=1}^{10} = (2, 4, 6, 8, 10, 2, 4, 6, 8, 10)?$$

Find a most likely path $(X_t^*)_{t=1}^{10}$ given the observations U_t above using Viterbi algorithm. Assume that the distribution of X_1 is uniform.