Markov Chains (273023), Exercise session 5, Tue 12 Feb 2013.

Exercise 5.1. Let $\Omega = \{1, 2, 3\}$ and $\mu = (0, 1/2, 1/2)$ and $\nu = (2/3, 0, 1/3)$. Find an optimal coupling (X, Y) for the measures μ and ν by explicitely giving the joint probability distribution and verifying that

$$\|\mu - \nu\|_{\mathrm{TV}} = \mathbb{P}(X \neq Y).$$

Exercise 5.2. Suppose we have two fair dice and we throw both. Let X be the value of the first die and Z the value of the second. If $Z \neq X$, then define $Y = Z$. If $Z = X$, then throw the second die again and let Y be the outcome. What is the total variation distance of the marginal distributions of X and Y? Are X and Y independent?

Exercise 5.3. Let P be a transition probability matrix of a Markov chain with N states. Assume that for all $t = 1, \ldots, 2N$ there is $x \in \Omega$ such that $P^t(x, x) = 0$. In addition, assume that there exists $T > 2N$ such that $P^T(x, x) > 0$ for all $x \in \Omega$. Show that P is reducible.

Addendum: The statement is false! Give an example of an irreducible P for which the above conditions hold.

Exercise 5.4. Let P be a transition probability matrix of a Markov chain X_t. Show that for every $x \in \Omega$ there exists $N > 0$ such that for every $j = 0, 1, \ldots, N-1$ there exists a unique probability distribution $\pi_{x,j}$ satisfying

$$\pi_{x,j}(y) = \lim_{k \to \infty} \mathbb{P}(X_{Nk+j} = y | X_0 = x).$$

Compare this result with exercise 2.2.

Hint: Use the convergence theorem for irreducible aperiodic Markov chains.

Exercise 5.5 (Levin, Peres, Wilmer: Ex. 12.1.(b), p. 167). Let P be the transition matrix of an irreducible Markov chain with finite state space Ω. Let

$$\mathcal{T}(x) = \{t > 0 : P^t(x, x) > 0\}.$$

Show that $\mathcal{T}(x) \subset 2\mathbb{Z}$ if and only if -1 is an eigenvalue of P.

Exercise 5.6 (Levin, Peres, Wilmer: Ex. 12.3., p. 168). Let P the transition probability matrix of a Markov chain. Let

$$\tilde{P} = \frac{P + I}{2}$$

where I is the identity matrix with ones on the diagonal and zeros elsewhere. Show that all the eigenvalues of $\tilde{P}$ are non-negative.