Markov Chains (273023), Exercise session 2, Tue 22 Jan 2013.

Exercise 2.1. Let P be a doubly stochastic matrix, i.e. all the row sums and the column sums equal to one. Show that the uniform distribution is a stationary distribution for the associated Markov chain.

Exercise 2.2. Let $(X_0, X_1, \dots)$ and $(Y_0, Y_1, \dots)$ be irreducible and aperiodic Markov chains with finite state spaces Ω_1, Ω_2 and transition probability matrices P_1, P_2. Let $(Z_0, Z_1, \dots)$ be a process defined on $\Omega = \Omega_1 \cup \Omega_2$ with a transition probability matrix

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}.$$

Let π_1 and π_2 be the unique stationary distributions of $(X_0, X_1, \dots)$ and $(Y_0, Y_1, \dots)$. Show that $(Z_0, Z_1, \dots)$ is a Markov chain and find its stationary distributions in terms of π_1 and π_2.

Exercise 2.3 (Levin, Peres, Wilmer: Ex. 1.2 p. 18). A graph G is *connected* when, for any two vertices x and y of G, there exists a sequence of vertices $x_0, x_1, \dots, x_k$ such that $x_0 = x$, $x_k = y$, and $x_i \sim x_{i+1}$ for all $0 \leq i \leq k-1$. Show that the simple random walk on G is irreducible if and only if G is connected.

Exercise 2.4 (Levin, Peres, Wilmer: Ex. 1.11 p. 19). Let P be a transition probability matrix of a finite Markov chain. Let μ be the uniform probability distribution over the state space. Define

$$\nu_n = \frac{1}{n} \sum_{k=0}^{n-1} \mu P^k$$

for $n = 1, 2, \dots$. Show that the sequence ν_n is bounded and hence there exists a converging subsequence with limit ν. Further show that ν is a stationary distribution.

Exercise 2.5 (Levin, Peres, Wilmer: Ex. 1.15 p. 19). Let A be a subset of a finite state space Ω of a Markov chain. Define

$$f(x) = \mathbb{E}(\tau_A, X_0 = x)$$

where

$$\tau_A = \min\{t \geq 0 : X_t \in A\}.$$

Show that $f(x) = 0$ for every $x \in A$ and

$$f(x) = 1 + \sum_{y \in \Omega} P(x, y) f(y)$$

for every $x \in \Omega \setminus A$. Conclude that f is uniquely determined by the above equations. This method is called First Step Analysis.

Exercise 2.6 (from the final exam of the Spring 2011). Let

$$P = \begin{pmatrix} .3 & .2 & 0 & 0 & .5 & 0 \\ 0 & .6 & .4 & 0 & 0 & 0 \\ 0 & .4 & .6 & 0 & 0 & 0 \\ .1 & .1 & .1 & .6 & 0 & .1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

on the state space $\Omega = \{1, 2, 3, 4, 5, 6\}$. Let $A \subset \Omega$ be the union of the absorbing classes. Assume that $X_0 = 4$. Using the result of the previous exercise, compute $\mathbb{E}(\tau_A, X_0 = 4)$.