Home Assignment 2

Markov Chains (273023), Return by 15 March. The purpose of the exercise is to analyze the behaviour of a complicated Markov chain.

To complete the assignment, please return a written report (max. 2 pages) by 15 March, as a pdf file by email to the lecturer.

The calculations can be done by computer or by hand. Only the main points should be included in the report, e.g. the fundamental matrix N can be given without showing how to invert $I - Q$.

NB: Only one pdf file for each group (1-2 persons). The pdf should be named: $surname_1 surname_2 pi.pdf$.

The task

Let

$$P = \begin{pmatrix} 0.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0.7 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1 & 0 & 0 & 0.9 \\ 0 & 0 & 0 & 0.2 & 0 & 0.2 & 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0.2 & 0 & 0 & 0 & 0 & 0.1 & 0.1 & 0 & 0.6 \end{pmatrix}$$

be the transition probability matrix of a Markov chain. Answer the following questions:

- Q1: What is the period of the states 1, 2 and 3?
- Q2: Does the chain has a stationary distribution? If it has, find one.
- Q3: What is the expected number of times the chain visits each state if $X_0 = 10$?
- Q4: What is the expected time for the chain to hit an absorbing class if $X_0 = 1$?
- Q5: Determine $\lim_{t\to\infty} P^t(10, 8)$.
- Q6: Determine $\lim_{t\to\infty} P^{2t+1}(1, 2)$.

A suggestion for implementation

Classify the states into transient and absorbing classes.

Reorganize the chain so that the transient states are first and each absorbing class has only one state to have a modified Markov chain $\tilde{X}_t$ with state space $\tilde{\Omega}$ and transition probability matrix

$$\tilde{P} = \begin{pmatrix} Q & R \\ 0 & I \end{pmatrix}$$

where I is the identity.

The periods of the states are defined as the greatest common divisor of

$$\mathcal{T}(x) = \{t > 0 : P^t(x, x) > 0\}.$$

If $\mathcal{T}(x)$ is empty, then the period is infinity.

The matrix $I - Q$ is invertible with inverse N, the so called fundamental matrix. The expected number of times the chains visists each state and the absorption times can be calculated with the help of N. For more details, study the book of Grinstead and Snell (with link on the course home page), Chapter 11, pp. 417–420.

For Q5 (and similarly for Q6), find first the unique stationary distribution of the absorbing class where the state 8 belongs to (observe that the subchain, i.e. the original chain restricted to the absorbing class, is irreducible and aperiodic). Second, calculate the probability of absorption to the class with 8. And finally combine the probabilities.