Chapter V

Multiresolution Analysis (MRA)

In this chapter we develop a general theory which resembles the Haar theory in Chapters III-IV, but which uses "better" classes of wavelets than the Haar wavelet.

V.1 Orthonormal Systems of Translates

Definition 1.1. The *translation* operator T_k is the shift by the amout k:

$$(T_k g)(x) = g(x-k), \quad x \in \mathbb{R},\ k \in \mathbb{Z},$$

(right-shift if $k > 0$, left-shift if $k < 0$).

Recall from Theorem 1.6, page 25: There we had $V_0 =$ dyadic step functions in $L^2(\mathbb{R})$ which are constant in each interval $[k, k+1)$, $k \in \mathbb{Z}$. An orthonormal basis for V_0 was given by $\{T_k p\}_{k=-\infty}^{\infty}$ (all integer translates of the Haar scaling function p on page 25). Now we look at *other* functions g with a similar property.

Definition 1.2. An orthonormal system of functions g_k of the type $g_k = T_k g$ (for some fixed $g \in L^2(\mathbb{R})$) is called an *orthonormal system of translates.*

Orthonormal means: $g_n \perp g_m$, $n \neq m$, $\|g_n\| = 1$.

Note. Need *not* be a *basis*: There will always exist functions $f \in L^2(\mathbb{R})$ which are orthonormal to *all* g_k.

Example 1.3. a) The collection of scale zero *Haar scaling functions* $\{p_{0,k}\}_{k=-\infty}^{\infty}$ is an orthonormal system of translates.

b) The collection of scale zero *Haar wavelets* $\{h_{0,k}\}_{k=-\infty}^{\infty}$ is another orthonormal system of translates.

Neither is a *basis* in $L^2(\mathbb{R})$.

Theorem 1.4. *Let $\{g_k\}_{k=-\infty}^{\infty}$ be an orthonormal system.*

(i) The set V of all functions $f \in L^2(\mathbb{R})$ which has an expansion of the type

$$f = \sum_{k=-\infty}^{\infty} c_k g_k, \quad \textit{where} \sum_{k=-\infty}^{\infty} |c_k|^2 < \infty, \tag{1}$$

is a closed supspace of $L^2(\mathbb{R})$.

(ii) $\{g_k\}_{k=-\infty}^{\infty}$ is an orthonormal basis for V.

(iii) The orthonormal projection of $L^2(\mathbb{R})$ onto V is given by

$$Pf = \sum_{k=-\infty}^{\infty} \langle f, g_k \rangle g_k.$$

Proof.

(i) See Analysis II.

(ii) If $f \perp$ every g_k and (1) holds, then

$$\begin{aligned} 0 &= \langle f, g_k \rangle = \langle \sum_{l=-\infty}^{\infty} c_l g_l, g_k \rangle \\ &= \sum_{l=-\infty}^{\infty} c_l \underbrace{\langle g_l, g_k \rangle}_{\delta_l^k} = c_k. \end{aligned}$$

Thus, $c_k = 0$ for all k. Substitute this in (1) $\implies f = 0$. By Definition 2.1 on page 12, $\{g_k\}_{k=-\infty}^{\infty}$ is a basis for V.

(iii) See Theorem 3.7 on page 14. □

Note. (For readers of Analysis II): $V =$ closed linear span of $\{g_k\}_{k=-\infty}^{\infty} = ((\{g_k\}_{k=-\infty}^{\infty})^{\perp})^{\perp}$

Theorem 1.5. *The collection* $\{T_k g\}_{k=-\infty}^{\infty}$ *is an orthonormal system* if *and* only if

$$\sum_{k=-\infty}^{\infty} |\hat{g}(\omega+k)|^2 = 1 \quad a.e.$$

Proof. Note first that

$$\begin{aligned}
\langle T_k g, T_l g\rangle &= \int_{-\infty}^{\infty} g(x-k)\overline{g(x-l)}dx \quad (x-l=y) \\
&= \int_{-\infty}^{\infty} g(y-k+l)\overline{g(y)}dy \\
&= \langle T_{k-l} g, g\rangle,
\end{aligned}$$

so that

$$\langle T_k g, T_l g\rangle = \delta_k^l \iff \langle T_k g, g\rangle = \delta_0^k$$

for all k. By Parsevals formula (see page 16 and 18)

$$\begin{aligned}
\langle T_k g, g\rangle &= \int_{\mathbb{R}} g(x-k)\overline{g(x)}dx \\
&= \int_{\mathbb{R}} \hat{g}(\omega)e^{-2\pi i k\omega}\overline{\hat{g}(\omega)}d\omega \\
&= \int_{\mathbb{R}} |\hat{g}(\omega)|^2 e^{-2\pi i k\omega}d\omega \\
&= \sum_{n=-\infty}^{\infty}\int_n^{n+1} |\hat{g}(\omega)|^2 e^{-2\pi i k\omega}d\omega \quad (\omega = n+\nu) \\
&= \sum_{n=-\infty}^{\infty}\int_0^{1} |\hat{g}(\nu+n)|^2 e^{-2\pi i k\nu}d\nu \\
&\quad \text{(Note that } e^{2\pi i k n = 1}\text{)} \\
&= \int_0^1 \Big(\sum_{n=-\infty}^{\infty} |\hat{g}(\nu+n|^2\Big)e^{-2\pi i k\nu}d\nu
\end{aligned}$$

(O.k. to change order since everything converges absolutely). The function $F(\nu) = \sum_{n=-\infty}^{\infty}|\hat{g}(\nu+n)|^2$ is a periodic function in $L^1(\mathbb{T})$ (easy to check). The k:th Fourier coefficient of this series is

$$\hat{F}(k) = \int_0^1 F(\nu)e^{-2\pi i k\nu}d\nu = \langle T_k g, g\rangle_{L^2(\mathbb{R})} \quad \text{(by the computation above)}. \tag{2}$$

If $F(\nu) \equiv 1$, then it is easy to check (by direct computation) that the inverse transform $\check{F}$ of F is given by

$$\check{F}(k) = \delta_0^k = \begin{cases} 1, & k = 0, \\ 0, & k = 0. \end{cases}$$

By (2) this implies that

$$\langle T_k g, g \rangle = \delta_0^k,$$

so $\{T_k g\}$ is an orthonormal system of translates. Conversely, if $\langle T_k g, g \rangle = \delta_0^k$, then the reconstruction formula for F gives, for almost all x,

$$\begin{aligned} F(x) &= \sum_{k=-\infty}^{\infty} \hat{F}(k) e^{2\pi i k x} \\ &= \sum_{k=-\infty}^{\infty} \delta_0^k e^{2\pi i k x} \equiv 1. \quad \square \end{aligned}$$

Let $\{T_k g\}_{k=-\infty}^{\infty}$ be an orthonormal system of translates. Is there an "easy" way to check if $f \in V$, where V is the subspace of $L^2(\mathbb{R})$ described in Theorem 1.4 with $g_k = T_k g$? Answer:

Theorem 1.6. *Let $\{T_k g\}_{k=-\infty}^{\infty}$ be an orthonormal system of translates, and let V be the subspace of $L^2(\mathbb{R})$ described in Theorem 1.4, page 42, with $g_k = T_k g$. Then $f \in V$ if and only if*

$$\hat{f}(\omega) = \hat{c}(\omega)\hat{g}(\omega),$$

where $\hat{c}(\omega)$ is a periodic function with period 1 satisfying $\int_0^1 |\hat{c}(\omega)|^2 d\omega < \infty$. The connection between this function and the coefficients c_k in the expansion

$$f = \sum_{k=-\infty}^{\infty} c_k T_k g$$

is the following: $\hat{c}$ is the discrete Fourier transform of the sequence $\{c_k\}_{k=-\infty}^{\infty}$ (see Section II.6), i.e.,

$$\begin{aligned} \hat{c}(\omega) &= \sum_{k=-\infty}^{\infty} e^{-2\pi i k \omega} c_k, \quad \textit{and} \\ c_k &= \int_0^1 e^{2\pi i k \omega} \hat{c}(\omega) d\omega. \end{aligned}$$

We have

$$\|f\|^2_{L^2(\mathbb{R})} = \|\hat{c}\|^2_{L^2(\mathbb{T})} = \|c\|^2_{\ell^2(\mathbb{Z})},$$

i.e.,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_0^1 |\hat{c}(\omega)|^2 d\omega = \sum_{k=-\infty}^{\infty} |c_k|^2,$$

and

$$c_k = \langle f, T_k g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x-k)}dx.$$

Moreover, if $f_1 \in V$ *has the expansion* $f_1 = \sum_{k=-\infty}^{\infty} c_{1,k} T_k g$ *and* $f_2 \in V$ *has the expansion* $f_2 = \sum_{k=-\infty}^{\infty} c_{2,k} T_k g$*, then*

$$\langle f_1, f_2\rangle_{L^2(\mathbb{R})} = \langle c_1, c_2\rangle_{\ell^2(\mathbb{Z})} = \langle \hat{c}_1, \hat{c}_2\rangle_{L^2(\mathbb{T})}$$

i.e.,

$$\int_{-\infty}^{\infty} f_1(\omega)\overline{f_2(\omega)}d\omega = \sum_{k=-\infty}^{\infty} c_{1,k}\overline{c_{2,k}} = \int_0^1 \hat{c}_1(\omega)\overline{\hat{c}_2(\omega)}d\omega.$$

Proof. Suppose that $f \in V$. First look at the case where f is a finite sum of the type

$$f(x) = \sum_{k=-N}^{N} c_k g(x-k).$$

Transforming this equation and using rule (a) on page 15 we get

$$\hat{f}(\omega) = \sum_{k=-N}^{N} c_k e^{-2\pi i\omega k}\hat{g}(\omega) = \hat{c}_N(\omega)\hat{g}(\omega)$$

where

$$\hat{c}_N(\omega) = \sum_{k=-N}^{N} c_k e^{-2\pi i\omega k}.$$

In the general case we let $N \to \infty$. Then

$$f_N(x) = \sum_{k=-N}^{N} c_k g(x-k) \to f(x) = \sum_{k=-\infty}^{\infty} c_k g(x-k)$$

in L^2 so $\hat{f}_N \to \hat{f}$ in $L^2(\mathbb{R})$. Likewise

$$\hat{c}_N(\omega) \to \hat{c}(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{-2\pi i \omega k}$$

in $L^2(0,1)$. Therefor $\hat{f}(\omega) = \hat{c}(\omega)\hat{g}(\omega)$ also in this case.

The proof of the converse is similar: Assume that $\hat{f}(\omega) = \hat{c}(\omega)\hat{g}(\omega)$ for some *periodic* $\hat{c}$ in $L^2(\mathbb{T})$. This function can be expanded in a Fourier series

$$\hat{c}(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{-2\pi i \omega k},$$

where

$$c_k = \int_0^1 e^{2\pi i \omega k} \hat{c}(\omega) d\omega.$$

Define

$$f_N(x) = \sum_{k=-N}^{N} c_k g(x-k),$$

and continue as above.

That

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{k=-\infty}^{\infty} |c_k|^2$$

follows from Bessel's equality (see Theorem 2.3 (iii) on page 12), since $\{T_k g\}_{k=-\infty}^{\infty}$ is an orthonormal basis for V (see Theorem 1.4 (iii) on page 42). By Plancherel's identity (see page 20)

$$\sum_{k=-\infty}^{\infty} |c_k|^2 = \int_0^1 |\hat{c}(\omega)|^2 d\omega.$$

That $c_k = \langle f, T_k g \rangle$ is a special case of Theorem 2.3 (iii) on page 12.

The final claim about the inner products follow from the fact that the operator which maps f into c is isometric (i.e., it preserves norms), and therefore, it also preserves inner products (this follows from the polarization identity). □

V.2 Non-orthogonal Systems of Translates

If we start with an arbitrary function $g \in L^2(\mathbb{R})$, then in most cases it is *not true* that $T_k g \perp g$. However, if g is "narrow enough" then it will be "almost true" in the following sense.

Recall: $\{T_k g\}_{k=-\infty}^{\infty}$ is an orthogonal system of translates $\iff \sum_{k=-\infty}^{\infty} |\hat{g}(\omega+k)|^2 = 1$.

Definition 2.1. Let $g \in L^2(\mathbb{R})$. The collection $\{T_k g\}_{k=-\infty}^{\infty}$ is a *Riesz system of translates* if there exists constants $0 < A \leq B < \infty$ such that

$$(0 <)A \leq \sum_{k=-\infty}^{\infty} |\hat{g}(\omega + k)|^2 \leq B(< \infty)$$

for (almost) all $\omega \in \mathbb{R}$.

Note 2.2. Every orthogonal system of translates is also a Riesz system (take $A = B = 1$).

Note 2.3. If $A = B$, then we get an orthogonal system of translates by replacing g by $g/\sqrt{A}$.

Proof. Define $\tilde{g} = g/\sqrt{A}$. Then

$$\sum_{k=-\infty}^{\infty} |\hat{\tilde{g}}(\omega + k)|^2 = \frac{1}{A} \sum_{k=-\infty}^{\infty} |\hat{g}(\omega + k)|^2 = 1.$$

The rest follows from Theorem 1.5 on page 43. □

Theorem 2.4. *Let $\{T_k g\}_{k=-\infty}^{\infty}$ be a Riesz system of translates, and let $\tilde{g}$ be the inverse Fourier transform of the function*

$$\hat{\tilde{g}}(\omega) = \frac{\hat{g}(\omega)}{\sqrt{\sum_{k=-\infty}^{\infty} |\hat{g}(\omega + k)|^2}}.$$

Then $\{T_k \tilde{g}\}_{k=-\infty}^{\infty}$ is an orthonormal system of translates.

Proof. Denote $F(\omega) := (\sum_{k=-\infty}^{\infty} |\hat{g}(\omega + k)|^2)^{1/2}$. Then $0 < A \leq F(\omega) \leq B < \infty$ for all ω (see page 47). We know that $\hat{g} \in L^2(\mathbb{R})$ since $g \in L^2(\mathbb{R})$ (by Plancherels identity). Therefore, also $\frac{\hat{g}(\omega)}{F(\omega)}$ belongs to $L^2(\mathbb{R})$ (it is $\leq \frac{|\hat{g}(\omega)|}{A}$).

This means that $\frac{\hat{g}(\omega)}{F(\omega)}$ has an inverse Fourier transform in $L^2(\mathbb{R})$, which we call $\tilde{g}$. Obviously

$$\begin{aligned}\sum_{k=-\infty}^{\infty} |\hat{\tilde{g}}(\omega+k)|^2 &= \sum_{k=-\infty}^{\infty} \frac{|\hat{g}(\omega+k)|^2}{|F(\omega+k)|^2} \\ &\quad \text{(The function } F \text{ is periodic with period 1, so } F(\omega+k)=F(\omega)\text{)} \\ &= \frac{1}{|F(\omega)|^2} \underbrace{\sum_{k=-\infty}^{\infty} |\hat{g}(\omega+k)|^2}_{=|F(\omega)|^2} = 1.\end{aligned}$$

By Theorem 1.5 on page 43, $\{T_k\tilde{g}\}$ is an orthonormal system of translates. $\square$

With the help of the orthogonalized functions $\tilde{g}$ we can define the space V as in, e.g., Theorem 1.6 on page 44. Can we say anything about how the *original* function g is related to this subspace?

Lemma 2.5. *In the situation described in Theorem 2.4, the function g lies in the subspace V spanned by $\{V_k\tilde{g}\}_{k=-\infty}^{\infty}$. More precisely,*

$$g(x) = \sum_{k=-\infty}^{\infty} c_k \tilde{g}(x-k)$$

where c_k is the inverse Fourier series of the function

$$\hat{c}(\omega) = \sqrt{\sum_{k=-\infty}^{\infty} |\hat{g}(\omega+k)|^2}.$$

Proof. This follows from Theorem 1.6 on page 44 and Theorem 2.4 on page 47. $\square$

Lemma 2.6. *There also exist coefficients e_k with $\sum_{k=-\infty}^{\infty}|e_k|^2 < \infty$ such that*

$$\tilde{g} = \sum_{k=-\infty}^{\infty} e_k g(x-k).$$

More precisely, e_k is the (inverse) Fourier series of the function

$$\hat{e}(\omega) = \frac{1}{\sqrt{\sum_{k=-\infty}^{\infty}|\hat{g}(\omega+k)|^2}}.$$

Proof. "Essentially" the same as the proof of Theorem 1.6: The algebraic computations remains the same (it does not matter that $\{T_k g\}$ is not orthogonal). However, the convergence proofs become less trivial in this case. (More details later if we have the time). $\square$

Lemma 2.7. *In the situation described in Theorem 2.4 and Lemmas 2.5-2.6, a function f belongs to V if and only if it can be written in the form*

$$f(x) = \sum_{k=-\infty}^{\infty} e_k g(x-k)$$

for some sequence $\{e_k\} \in \ell^2(\mathbb{Z})$ (i.e., $\sum_k |e_k|^2 < \infty$). More precisely, the relationship between the coefficients e_k above and the coefficients c_k in the expansion

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \tilde{g}(x-k)$$

is the following: The Fourier transforms of $\{c_k\}$ and $\{e_k\}$ satisfy

$$\hat{e}(\omega) = \frac{\hat{c}(\omega)}{\sqrt{\sum_{k=-\infty}^{\infty} |\hat{g}(\omega+k)|^2}}.$$

(Thus, we get $\{e_k\}$ from $\{c_k\}$ by first taking a Fourier transform, then dividing with the square root, and finally taking the inverse Fourier transform).

Proof. Same comments as in the proof of Lemma 2.6. $\square$

Corollary 2.8. *The following are equivalent:*

(i) $f \perp V$ *(i.e., $f \perp g$ for all $g \in V$)*

(ii) $f \perp T_k \tilde{g}$ *for all k*

(iii) $f \perp T_k g$ *for all k.*

Proof. Follows from Lemma 2.7. $\square$

V.3 Multiresolution Analysis

In the preceeding sections we only looked at *translations* of a fixed function g. Now we also introduce *dilations* (and *compressions*).

Definition 3.1. The *dilation operator* D_a (for all $a > 0$) is given by

$$(D_a g)(x) = \sqrt{a} g(ax), \quad x \in \mathbb{R}$$

(dilation for $0 < a < 1$, compression for $a > 1$).

Definition 3.2. A multiresolution analysis (MRA) on $\mathbb{R}$ is a sequence of closed subspaces $\{V_j\}_{j=-\infty}^{\infty}$ of functions in $L^2(\mathbb{R})$ with the following properties:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,

(b) $\bigcap_{j=-\infty}^{\infty} V_j = 0$,

(c) $\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(\mathbb{R})$,

(d) $f \in V_0$ if and only if $D_{2^j} f \in V_j$,

(e) There is a function $\varphi \in L^2(\mathbb{R})$, called a *scaling function* such that $\{T_k \varphi\}_{k=-\infty}^{\infty}$ is an orthogonal system of translates, and such that $f \in V_0$ if and only if it has an expansion of the type

$$f = \sum_{k=-\infty}^{\infty} c_k T_k \varphi$$

(Compare this to Theorem 1.4, page 42). (Note: φ is also called "father wavelet").

Note 3.3.

(i) Some people like to write (a)-(c) in the form

$$\{0\} \leftarrow \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots \rightarrow L^2(\mathbb{R})$$

(intersection is $\{0\}$, closure of union is $L^2(\mathbb{R})$).

(ii) Also note that we get from V_j to V_{j+1} by *compressing* the function by a factor 2, and from V_{j+1} to V_j by *dilating* the function by a factor two.

(iii) It follows from Theorem 1.4 (page 42) and Theorem 2.3 (page 12), that for all $f \in V_0$ we have

$$\|f\|_{L^2(\mathbb{R})} = \sum_{k=-\infty}^{\infty} |\langle T_k\varphi, f\rangle|^2.$$

Assuming properties (a)-(e) in Definition 3.2, we may now proceed exactly like in the Haar case.

Definition 3.4. For each $j, k \in \mathbb{Z}$, we define $\varphi_{j,k}$ by

$$\varphi_{j,k}(x) = 2^{j/2}\varphi(2^j x - k) = (D_{2^j}T_k\varphi)(x).$$

For each $j \in \mathbb{Z}$ we define the *approximation operator* P_j by

$$P_j f = \sum_{k=-\infty}^{\infty} \langle f, \varphi_{j,k}\rangle\varphi_{j,k},$$

and the *detail operator* Q_j by

$$Q_j f = P_{j+1}f - P_j f.$$

Lemma 3.5. *For each $j \in \mathbb{Z}$, $\{\varphi_{j,k}\}_{k=-\infty}^{\infty}$ is an orthonormal basis for V_j.*

Proof. By Definition 3.2 (e), each $\varphi(x-k) = T_k\varphi$ belongs to V_0, and by property (d), we have

$$\varphi_{j,k} = D_{2^j}T_k\varphi \in V_j.$$

To see that it is an orthonormal system in V_j we compute

$$\begin{aligned}\langle\varphi_{j,k}, \varphi_{j,m}\rangle &= \int_{-\infty}^{\infty} 2^{j/2}\varphi(2^j x - k)2^{j/2}\overline{\varphi(2^j x - m)}dx \quad (2^j x - m = y) \\ &= \int_{-\infty}^{\infty} \varphi(y + m - k)\overline{\varphi(y)}dy = \delta_0^{k-m}.\end{aligned}$$

Thus, it is orthonormal. By using a similar change of variable we can get an expansion for an arbitrary $f \in V_j$: Since $D_{2^{-j}}f \in V_0$, we have an expansion for $D_{2^{-j}}f$:

$$D_{2^{-j}}f = \sum_{k=-\infty}^{\infty} \langle D_{2^{-j}}f, T_k\varphi\rangle\varphi_{j,k}$$

and finally make a change of variable as above (check this) to get

$$\langle D_{2^{-j}}f, T_k\varphi\rangle = \langle f, D_{2^j}T_k\varphi\rangle = \langle f, \varphi_{j,k}\rangle.$$

By Theorem 2.3 on page 12 $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for V_j. □

Lemma 3.6. *For all $f \in L^2(\mathbb{R})$,*

(a) $P_jf \to f$ *as* $j \to \infty$,

(b) $P_jf \to 0$ *as* $j \to -\infty$.

Proof.

(a) Same as proof of Theorem 3.2 iii) on page 31.

(b) Define $R_j = I - P_j$. Then R_j is the orthogonal projection onto $X_j = L^2(\mathbb{R}) \ominus V_j = V_j^\perp$. It follows from conditions (a)-(c) that

(a1) $R_{j+1} \subset R_j$

(b1) $\bigcap_{j=-\infty}^{\infty} X_j = \bigcap_{j=-\infty}^{\infty} V_j^\perp = (\bigcup_{j=-\infty}^{\infty} V_j)^\perp$

(c1) $\overline{\bigcup_{j=-\infty}^{\infty} X_j} = ((\bigcup_{j=-\infty}^{\infty} X_j)^\perp)^\perp = (\bigcap_{j=-\infty}^{\infty} X_j^\perp)^\perp = (\bigcap_{j=-\infty}^{\infty} V_j)^\perp = \{0\}^\perp = L^2(\mathbb{R})$.

Thus, by repeating the proof of (a) with the replacements $V_j \to X_j$, $P_j \to R_j$ and letting $j \to -\infty$ instead of $j \to +\infty$, we find that $R_jf \to f$ as $j \to -\infty$ for all $f \in L^2(\mathbb{R})$. Therefore $Pjf = (I - R_j)f = f - R_jf \to 0$ as $j \to -\infty$. □

Lemma 3.7. *Condition (b) in Definition 3.2 is redundant, i.e., it follows from conditions (d) and (e).*

Proof. Above we proved part (b) of Lemma 3.6 by using conditions (a) and (c) in Definition 3.2. It is also possible to prove part (b) of Lemma 3.6 by instead using only conditions (d) and (e) in Definition 3.2. See Walnut's book or Gripenberg's notes. Thus, (d) and (e) in Definition 3.2 imply (b) in Lemma 3.6. It is easy to see that (b) in Lemma 3.6 implies (b) in Definition 3.2. □

The following result can be used to check condition (c) in Definition 3.2:

Lemma 3.8. *Let (d) and (e) in Definition 3.2 hold, and suppose that at least one of the following conditions hold:*

(A) $\lim_{\substack{T\to\infty \\ S\to-\infty}} \int_S^T \varphi(x)dx = \gamma$ *where* $|\gamma| = 1$

(B) $\hat{\varphi}$ *is continuous at zero, and* $\hat{\varphi}(0) = 1$.

Then condition (c) in Definition 3.2 holds.

Proof. See Gripenberg's notes. A partial converse to this statement is found in Lemma **??** below.

Thus, if conditions (d) and (e) hold, then (c) is "almost equivalent" to $|\hat{\varphi}(0)| = 1$. □

By using condition (d) in Definition 3.2 on page 50 we get the important

Theorem 3.9 (Two-scale Dilation *relations*)**.** *Assume that (a) - (e) of Definition 3.2 hold. Define the sequence* $\alpha(k)$ *by*

$$\alpha(k) = \int_{-\infty}^{\infty} \varphi(x)\overline{\varphi(2x-k)}dx = 2^{-1/2}\langle \varphi, \varphi_{1,k} \rangle. \tag{3}$$

Then

$$\varphi(x) = 2\sum_{k=-\infty}^{\infty} \alpha(k)\varphi(2x-k), \tag{4}$$

and

$$\hat{\varphi}(2\omega) = \hat{\alpha}(\omega)\hat{\varphi}(\omega), \tag{5}$$

where $\hat{\alpha}(\omega) = \sum_{k=-\infty}^{\infty} e^{-2\pi i\omega k}\alpha(k)$ *is the Fourier transform of* $\{\alpha(k)\}_{k=-\infty}^{\infty}$. *Moreover*

$$\sum_{k=-\infty}^{\infty} |\alpha(k)|^2 = \frac{1}{2}, \tag{6}$$

and

$$|\hat{\alpha}(\omega)|^2 + |\hat{\alpha}(\omega+\frac{1}{2})|^2 = 1, \quad \omega \in \mathbb{R}. \tag{7}$$

Note. $\hat{\alpha}(\omega)$ is periodic with period 1.

Note. Here we use the same scaling factor as in Gripenberg's notes. Walnut replaces the factor 2 in (4) by $\sqrt{2}$. This leads to a number of factors $\sqrt{2}$ in all the remaining formulas.

Proof. Since $\varphi \in V_0 \subset V_1$ and $\varphi_{1,k}$ is an orthonormal basis for V_1 (see Lemma 3.5, page 51), it follows that

$$\begin{aligned}
\varphi(x) &= \sum_{k=-\infty}^{\infty} \langle \varphi, \varphi_{1,k} \rangle \varphi_{1,k} \\
&= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(y)\sqrt{2}\ \overline{\varphi(2y-k)} dy\ \sqrt{2}\varphi(2x-k) \\
&= 2 \sum_{k=-\infty}^{\infty} \alpha(k)\varphi(2x-k).
\end{aligned}$$

This proves (4). Formula (6) follows from the fact that

$$1 = \|\varphi\|^2_{L^2(\mathbb{R})} = \sum_{k=-\infty}^{\infty} |\langle \varphi, \varphi_{1,k} \rangle|^2 = 2 \sum_{k=-\infty}^{\infty} |\alpha(k)|^2.$$

To derive (5) we first replace x by $\frac{x}{2}$ in (4) to get

$$2\varphi(\frac{x}{2}) = \sum_{k=-\infty}^{\infty} \alpha(k)\varphi(x-k),$$

and then we use Theorem 1.6 on page 44 and property (e) on page 17 to get

$$\hat{\varphi}(2\omega) = \hat{\alpha}(\omega)\hat{\varphi}(\omega).$$

The only remaining claim is (7). To get this we use Theorem 1.5 on page 43 (have replaced ω by 2ω):

$$
\begin{aligned}
1 &= \sum_{k=-\infty}^{\infty} |\hat{\varphi}(2\omega + k)|^2 \\
&= \sum_{k=-\infty}^{\infty} |\hat{\alpha}(\omega + \frac{k}{2})|^2 |\hat{\varphi}(\omega + \frac{k}{2})|^2 \quad \text{split into sum of odd and even } k \\
&= \sum_m |\hat{\alpha}(\omega + \frac{2m}{2})|^2 |\hat{\varphi}(\omega + \frac{2m}{2})|^2 \quad \text{(even)} \\
&\quad + \sum_m |\hat{\alpha}(\omega + \frac{2m+1}{2})|^2 |\hat{\varphi}(\omega + \frac{2m+1}{2})|^2 \quad \text{(odd)} \\
&\quad (\hat{\alpha} \text{ is periodic with period one} \\
&= |\hat{\alpha}(\omega)|^2 \sum_m |\hat{\varphi}(\omega + m)|^2 \\
&\quad + |\hat{\alpha}(\omega + \frac{1}{2})|^2 \sum_m |\hat{\varphi}(\omega + \frac{1}{2} + m)|^2 \\
&\quad \text{(use Theorem 1.5 again)} \\
&= |\hat{\alpha}(\omega)|^2 + |\hat{\alpha}(\omega + \frac{1}{2})|^2 \quad \square
\end{aligned}
$$

We call the sequence α in Theorem 3.9 a *filter* if, in addition, $\sum |\alpha(k)| < \infty$.

Definition 3.10. A sequence $\{\alpha(k)\}_{-\infty}^{\infty}$ is called a *quadrature mirror filter* if

A) $\alpha \in \ell^1$, i.e., $\sum |\alpha(k)| < \infty$ (sometimes omitted), and

B) $|\hat{\alpha}(\omega)|^2 + |\hat{\alpha}(\omega + \frac{1}{2})|^2 = 1$, (this is condition (7) on page 53).

The filter in $\{\alpha(k)\}_{k=-\infty}^{\infty}$ in Theorem 3.9 is called the **qudrature mirror filter induced by the multiresolution analysis in Definition 3.2** (assuming that $\sum |\alpha(k)| < \infty$).

V.4 Examples of MRA

V.4.1 The Haar MRA

Here $\varphi =$ the Haar scaling function

$$\varphi(x) = \begin{cases} 1, & 0 \leq x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

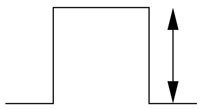


Figure V.1: The Haar scaling function.

$V_0 =$ collection of dyadic step functions in L^2 with step length one. This is a MRA. See Lemma 1.5 on page 24, Theorem 1.6 on page 25.

V.4.2 The Piecewise Linear MRA

Let V_j consist of all *continuous functions* in $L^2(\mathbb{R})$ which are *linear* in each dyadic interval of length 2^{-j}. Then it is easy to show that properties (a)-(d) in Definition 3.2 (page 50) holds (the computations are analogous to those in the Haar case. The only problematic condition is condition (e)).

We could try to use a triangular function $\varphi(x)$:

$$\varphi(x) = \begin{cases} 1 - |x|, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

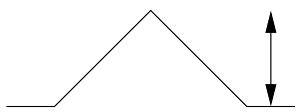


Figure V.2: The triangular function φ.

This is a *nice* function in the sense that it is obvious that every function in V_0 can be written as a sum of the type

$$f = \sum c_k T_k \varphi.$$

However, $\{T_k\varphi\}_{k=-\infty}^{\infty}$ is not an *orthogonal* system of translates. Fortunately, it is still a Riesz system of translates. Let us define a new function $\tilde{\varphi}$ as in Theorem 2.4 on page 47 through its Fourier transform:

$$\begin{aligned} \hat{\tilde{\varphi}}(\omega) &= \frac{\hat{\varphi}(\omega)}{(\sum_{k=-\infty}^{\infty}|\hat{\varphi}(\omega+k)|)^{1/2}} \\ &= \frac{\hat{\varphi}(\omega)}{\left(\frac{1}{3}(1+2\cos^2(\pi\omega))\right)^{1/2}}. \end{aligned}$$

This gives us an orthogonal system of translates (see Theorem 2.4 on page 47), and the space V_0 corresponding to $\tilde{\varphi}$ is *the same* space V_0 which we had above (see Lemma 2.7 on page 49).

We conclude that this is a MRA. The scaling function $\tilde{\varphi}$ is obtained from φ as a sum (see Lemma 2.6 on page 48)

$$\tilde{\varphi}(x) = \sum_{k=-\infty}^{\infty} e_k \varphi(x-k),$$

where $\{e_k\}_{k=-\infty}^{\infty}$ is the inverse Fourier transform of the periodic function

$$\hat{e}(\omega) = \frac{1}{\left(\frac{1}{3}(1+2\cos^2(\pi\omega))\right)^{1/2}}.$$

This function has *infinite support* (but it decays exponetially). See picture in Walnut on page 188.

V.4.3 The Shannon (bandlimited) MRA

Here we define φ through its Fourier transform:

$$\hat{\varphi}(\omega) = \begin{cases} 1, & |\omega| \le \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

By Theorem 1.5 on page 43, $\{T_k\varphi\}$ is an orthogonal system of translates. We define the space V_0 as in Theorem 1.6 on page 44. It contains all functions which has an expansion of the type

$$f \in V_0 \iff f = \sum_{k=-\infty}^{\infty} c_k T_k \varphi,$$

and analogously, we get V_j by replacing φ by $D_{2^j}\varphi$ (a suitable compression of φ). Then (d)-(e) in Definition 3.2 hold. To check that also (a)-(c) hold we look at the Fourier transforms of the functions in V_0 and V_j.

By Theorem 1.6 on page 44, $f \in V_0$ if and only if

$$\hat{f}(\omega) = 0 \text{ for } |\omega| > \frac{1}{2}, \text{ and } \int_{-1/2}^{1/2} |\hat{f}(\omega)|^2 d\omega < \infty.$$

Analogously, $f \in V_j$ if and only if

$$\hat{f}(\omega) = 0 \text{ for } |\omega| > \underbrace{\frac{1}{2}2^j}_{=2^{j-1}}, \text{ and } \int_{-2^{j-1}}^{2^{j-1}} |\hat{f}(\omega)|^2 d\omega < \infty.$$

From this follows that (a)-(c) hold.

Note. By inverting the Fourier transform in (8) we get

$$\varphi(x) = \frac{\sin(\pi x)}{\pi x}.$$

See picture in Walnut on page 189.

Note. The Shannon scaling function is *very poorly located* in time:

$$\int_{-\infty}^{\infty} |\varphi(x)| dx = \infty \quad (\iff \varphi \notin L^1(\mathbb{R}))$$

(although $\int_{-\infty}^{\infty} |\varphi(x)|^2 dx = 1 < \infty$).

V.4.4 The Strömberg and Meyer MRA

Essentially same result, done independently first by Strömberg and later by Meyer.

The idea is to use the same construction as in the Shannon case, but to replace the square window function

$$\hat{\varphi}(\omega) = \begin{cases} 1, & |\omega| \le \frac{1}{2}, \\ 0, & |\omega| > \frac{1}{2}. \end{cases}$$

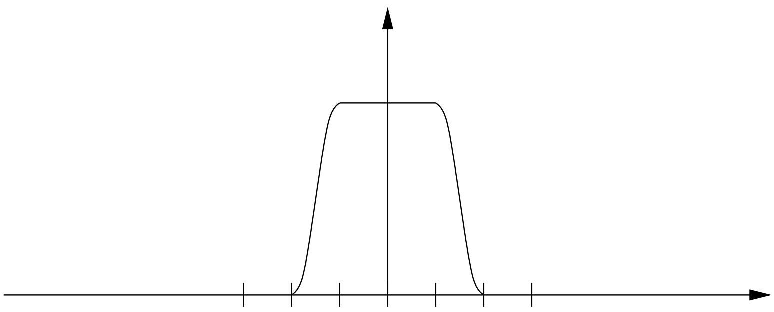
by a smoother function. We choose this function so that it satisfies the following conditions:

i) $\hat{\varphi}(\omega) = 0$ for $|\omega| \geq \frac{2}{3}$,

ii) $\hat{\varphi}(\omega) = 1$ for $|\omega| \leq \frac{1}{3}$,

iii) $0 \leq \hat{\varphi}(\omega) \leq 1$ for $\frac{1}{3} \leq |\omega| \leq \frac{2}{3}$, and

iv) $|\hat{\varphi}(\omega)|^2 + |\hat{\varphi}(\omega - 1)|^2 = 1$, for $\frac{1}{3} \leq |\omega| \leq \frac{2}{3}$.



Explicit formulas giving examples on how $\hat{\varphi}(\omega)$ should be defined for $\frac{1}{3} \leq \omega \leq \frac{2}{3}$ are given in just about all books on wavelets. We can make φ have any finite number of derivatives (as Strömberg did) or even make it C^∞ (as Meyer did). The proof that this gives us a MRA is almost the same as in the Shannon case. (Also here all functions in V_0 are bandlimited with band limit $\frac{2}{3}$, but *not every* f satisfying $|\hat{f}(\omega)| = 0$ for $|\omega| \geq \frac{2}{3}$ belongs to V_0. In addition $\hat{f}$ must be "small" as $\omega \to \pm\frac{2}{3}$). See picture on page 190 in Walnut.

V.5 The Detail Spaces W_j

Recall: In the Haar construction we first had the approximation spaces

$$\{0\} \leftarrow \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots \to L^2(\mathbb{R}).$$

Then we split each V_{j+1} into the ortogonal sum

$$V_{j+1} = V_j \oplus W_j,$$

and got a "sum decomposition" of $L^2(\mathbb{R})$:

$$L^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j \tag{9}$$

(where $W_j \perp W_l$ for $j \neq l$).

In addition we succeeded to find an *orthonormal basis* for W_j by dilating and translating the *mother wavelet* ψ.

Question: Given a general MRA of the type discussed in section V.4, let W_j be the orthogonal complement to V_j in V_{j+1}, so that

$$V_{j+1} = V_j \oplus W_j$$

A) Can we repeat the Haar construction and get (9)?

B) Can we find a "wavelet basis" for W_j?

What do we mean by a wavelet basis?

To answer these questions, let us proceed as in the Haar case (Chapter III).

Theorem 5.1. *The* approximation operators P_j *defined on page 51 have the following properties:*

(i) P_j *is the orthogonal projection of* $L^2(\mathbb{R})$ *onto* V_j.

(ii) $P_j f = f$ *if* $f \in V_J$ *for some* $J \leq j$.

(iii) $\lim_{j\to\infty} P_j f = f$ *for all* $f \in L^2(\mathbb{R})$.

(iv) $\lim_{j\to-\infty} P_j f = 0$ *for all* $f \in L^2(\mathbb{R})$.

(v) $P_j P_J f = P_J P_j f = P_J f$ *when* $J \leq j$.

Proof.

(i) See Theorem 3.7 on page 14, part (e) of Definition 3.2 (page 50), and Lemma 3.5, page 51.

(ii) We have $P_j f = f$ for all $f \in V_j$, (since P_j is a projection onto V_j), and $V_J \subset V_j$ if $J \leq j$.

(iii) See Lemma 3.6 on page 52.

(iv) See Lemma 3.6 on page 52.

(v) It follows from (ii) that $P_j P_J f = P_J f$. Let U be the orthogonal complement in V_j to V_J so that $V_j = V_J \oplus U$. Then every $f \in L^2(\mathbb{R})$ can be split into three parts:

First we split f into $f = g + h$ where $g = P_j f$, $h = f - g$. Then $g \in V_j$, and

$$P_j h = P_j(f - P_j f) = P_j f - P_j^2 f = P_j f - P_j f = 0,$$

so $P_j h = 0 \iff h \in \mathcal{N}(P_j) \iff h$ is orthogonal to V_j.

Next we split g into $g = g_1 + g_2$, where $g_1 \in V_J$ and $g_2 \in U$. We have $g_2 \in U$ and $h \in V_j^{\perp}$, so both g_2 and h are orthogonal to V_J $(\in V_j)$. Therefore $P_J g_2 = 0$ and $P_J h = 0$, so

$$P_J f = P_J(g_1 + g_2 + h) = P_J g_1 \ (= g_1) = P_J(g_1 + g_2) = P_J g = P_J P_j f.$$

Thus $P_J = P_J P_j$ as claimed. □

Theorem 5.2. *The* detail operators Q_j *defined on page 51 have the following properties:*

(i) Q_j *is the orthogonal projection of* $L^2(\mathbb{R})$ *onto the orthogonal complement* W_j *to* V_j *in* V_{j+1}*, so that* $V_{j+1} = V_j \oplus W_j$*. We call* W_j *the* detail space *of scale* 2^{-j}.

(ii) $P_{j+1} = P_j + Q_j$.

(iii) $P_j = P_J + \sum_{J \le l < j} Q_l$, $j > J$.

(iv) $\lim_{j\to\infty} Q_j f = 0$ *for all* $f \in L^2(\mathbb{R})$.

(v) $\lim_{j\to-\infty} Q_j f = 0$ *for all* $f \in L^2(\mathbb{R})$.

(vi) $Q_j Q_l f = 0$ *for all* $j \neq l$, $f \in L^2(\mathbb{R})$.

Note. Compare this to Theorem 4.2 on page 33.

Proof.

(i) Let us *define* $\tilde{Q}_j$ to be the orthogonal projection of $L^2(\mathbb{R})$ onto W_j. As in the proof of Theorem 5.1 we split an arbitrary $f \in L^2(\mathbb{R})$ into $f = f_1 + f_2 + f_3$ where

$$f_1 \in W_j, \quad f_2 \in V_j, \quad f_3 \in V_{j+1}.$$

Then (as in that proof)

$$\tilde{Q}_j f_1 = f_1, \quad \tilde{Q}_j f_2 = 0, \quad \tilde{Q}_j f_3 = 0$$

(since f_2 and f_3 are orthogonal to W_j)

$$P_j f = f_2, \quad P_j f_1 = 0, \quad P_j f_3 = 0,$$

$$P_{j+1} f = f_1 + f_2, \quad P_{j+1} f_3 = 0.$$

Thus:

$$P_{j+1} f = f_1 + f_2 = \tilde{Q}_j f + P_j f,$$

so

$$P_{j+1} = P_j + \tilde{Q}_j, \quad \text{and} \quad \tilde{Q}_j = P_{j+1} - P_j.$$

This is how we defined Q_{j+1} on page 51, so $Q_j = \tilde{Q}_j =$ orthogonal projection onto W_j.

(ii)-(v) These proofs are the *same* as the ones on page 33.

(vi) Follows from the fact that $W_j \perp W_l$ for all $j \neq l$. We have $Q_l f \in W_l$, and $W_l \perp W_j$, so $Q_j[Q_l f] = 0$. □

V.6 The Mother Wavelet ψ

To proceed we need a "mother wavelet":

Definition 6.1. Given a general MRA of the type discussed in Section V.4, with the approximation spaces V_j (see page 50) and detail spaces W_j (see page 61), we call $\psi \in V_1$ a *mother wavelet* if

$$\{T_k\psi\}_{k=-\infty}^{\infty} \text{ is an } \textit{orthonormal} \text{ basis for } W_0$$

(Recall: $\{T_k\psi\}_{k=-\infty}^{\infty}$ is an orthonormal basis for V_0; see Lemma 3.5, page 51).

How can we check if ψ is a mother wavelet?

Lemma 6.2. $\psi \in L^2(\mathbb{R})$ *is a mother wavelet if and only if*

(i) $\psi \in V_1$

(ii) $\{T_k\psi\}_{k=-\infty}^{\infty}$ *is an orthonormal system of translates*

(iii) $\langle T_k\psi, T_l\varphi\rangle = 0$ *for all* k, l *(where* φ *is the father wavelet = scaling function)*

(iv) No function $f \in W_0$ *is orthogonal to every* $T_k\psi$, $k \in \mathbb{Z}$.

Proof. Conditions (i), (ii) and (iii) are *necessary* for ψ to be a mother wavelet (by definition). Suppose that these conditions hold. Let W be the space spanned by the functions $\{T_k\psi\}$, as in Theorem 1.4 on page 42, i.e., every $f \in W$ is of the form $f = \sum_{k=-\infty}^{\infty} d_k T_k\psi$. Then by that theorem, $\{T_k\psi\}$ is an orthogonal basis for W. We claim that $W = W_0$.

Because of (i), $\psi \in V_1$, and since $T_k V_1 = V_1$ (why?), we have $T_k\psi \in V_1$ for all k. Thus $W \subset V_1$.

By (iii), $T_k\psi \perp T_l\varphi$ for all $l \implies T_k\psi \perp V_0$ (since $\{T_l\varphi\}$ is a basis for V_0)

$$\implies W \perp V_0.$$

We conclude that $W \subset W_0$ (since $V_1 = V_0 \oplus W_0$).

If $W \neq W_0$, then there is some $f \in W_0$, $f \neq 0$, such that $f \perp W$. But then, by (iv), $f \perp T_k\psi$ for all k, and so $f = 0$. This shows that we cannot have $W \neq W_0$.

Consequently, $\{T_k\psi\}_{k=-\infty}^{\infty}$ is an orthonormal basis for W_0 $(= W)$. □

By using the mother wavelet ψ (if it exists) we can repeat many of the theorems in the Haar chapter.

Definition 6.3. For each $j, k \in \mathbb{Z}$, we define the *wavelet family* $\psi_{j,k}$ induced by the mother wavelet ψ by

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k) = (D_{2^j} T_k \psi)(x)$$

(cf. page 51).

Lemma 6.4. *For each* $j \in \mathbb{Z}$, $\{\psi_{j,k}\}_{k=-\infty}^{\infty}$ *is an orthonormal basis for* W_j.

Proof. Same as proof of Lemma 3.5 on page 51. □

(Compare this to Theorem 2.4 on page 27.)

Theorem 6.5. *Suppose that we have a MRA with scaling function* φ *and mother wavelet* ψ*. Then* *(a) For every $j > J$, the set of functions*

$$\{\varphi_{J,k}\}_{k=-\infty}^{\infty} \bigcup \{\psi_{l,k}\}_{\substack{-\infty<k<\infty \\ J\le l<j}}$$

is an orthonormal basis for V_j.

(b) For every $J \in \mathbb{Z}$, the set of functions

$$\{\varphi_{J,k}\}_{k=-\infty}^{\infty} \bigcup \{\psi_{j,k}\}_{\substack{j\ge J \\ -\infty<k<\infty}}$$

is an orthonormal basis in $L^2(\mathbb{R})$.

(c) The set of functions

$$\{\psi_{j,k}\}_{j,k=-\infty}^{\infty}$$

is an orthonormal basis in $L^2(\mathbb{R})$.

Proof. Same as the proofs of Theorems 2.7-2.9 on pages 28-29. □

Question 6.6.

A) Does there always exist a mother wavelet ψ?

B) How do we find it?

To answer this question we first take a closer look at the functions in V_{-1} and W_0.

Lemma 6.7. *Let φ be a scaling function of a MRA $\{V_j\}_{j=-\infty}^{\infty}$, and let $\hat{\alpha}$ be the function in Theorem 3.9, page 53 (the "lowpass filter"). Then*

i) $f \in V_0 \iff \hat{f}(\omega) = \hat{c}(\omega)\hat{\varphi}(\omega)$ for some periodic $\hat{c} \in L^2(\mathbb{T})$.
In this case

$$\|f\|_{L^2}^2 = \int_0^1 |\hat{c}(\omega)|^2 d\omega = \sum_{k=-\infty}^{\infty} |c_k|^2,$$

where c_k are the coefficients in the expansion

$$f = \sum_{k=-\infty}^{\infty} c_k T_k \varphi.$$

ii) $f \in V_{-1} \iff \hat{f}(\omega) = \hat{l}(\omega)\hat{\alpha}(\omega)\hat{\psi}(\omega)$ *where* $\hat{l}(\omega)$ *is periodic with* period $\frac{1}{2}$, *and*

$$\|f\|_{L^2}^2 = \int_0^{1/2} |\hat{l}(\omega)|^2 d\omega.$$

Proof.

i) See Theorem 1.6 on page 44.

ii) We have $f \in V_{-1} \iff D_2 f \in V_0$, so by i),

$$\widehat{D_2 f}(\omega) = \hat{c}(\omega)\hat{\varphi}(\omega).$$

Here

$$\begin{aligned}
\widehat{D_{1/2} f}(\omega) &= \int_{-\infty}^{\infty} e^{-2\pi i\omega x}\sqrt{2} f(2x)dx \quad (2x = y) \\
&= \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-\pi i\omega y} f(y)dy \\
&= \frac{1}{\sqrt{2}}\hat{f}(\frac{\omega}{2}) = \hat{c}(\omega)\hat{\varphi}(\omega).
\end{aligned}$$

Replace $\omega/2$ by $\omega \implies$

$$\begin{aligned}
\hat{f}(\omega) &= \sqrt{2}\hat{c}(2\omega)\hat{\psi}(2\omega) \quad \text{(use Theorem 3.9, page 53)} \\
&= \sqrt{2}\hat{c}(2\omega)\hat{\alpha}(\omega)\hat{\psi}(\omega) \\
&= \hat{l}(\omega)\hat{\alpha}(\omega)\hat{\varphi}(\omega), \quad \text{where } \hat{l}(\omega) = \sqrt{2}\hat{c}(2\omega).
\end{aligned}$$

Finally

$$\begin{aligned}
\|f\|_{L^2}^2 &= \|D_{1/2} f\|_{L^2}^2 \quad \text{(use i))} \\
&= \int_0^1 |\hat{c}(\omega)|^2 d\omega \quad (\omega = 2\nu) \\
&= 2\int_0^{1/2} |\hat{c}(2\nu)|^2 d\nu = \int_0^{1/2} |\hat{l}(\nu)|^2 d\nu. \quad \square
\end{aligned}$$

Lemma 6.8. *With the same assumptions as in Lemma 2.6, we have* *iii)* $f \in W_{-1} \iff \hat{f}(\omega) = \lambda(\omega)\overline{\hat{\alpha}(\omega + \frac{1}{2})}\hat{\varphi}(\omega)$, *where* λ *is antiperiodic with period* $\frac{1}{2}$: $\lambda(\omega + \frac{1}{2}) = -\lambda(\omega)$, *and*

$$\|f\|_{L^2}^2 = \int_0^{1/2} |\lambda(\omega)|^2 d\omega \ (< \infty)$$

(not that this is $= \int_{1/2}^1 |\lambda(\omega)|^2 d\omega$*).*

Proof. $f \in W_{-1} \subset V_0$, so by Lemma 6.7,

$$\hat{f}(\omega) = \hat{c}(\omega)\hat{\varphi}(\omega) \tag{10}$$

for some 1-periodic $\hat{c}$. On the other hand, $f \perp V_{-1}$, so

$$\begin{aligned} &\langle g, f\rangle = 0 \quad \forall g \in V_{-1} && \iff \\ &\langle \hat{g}, \hat{f}\rangle = 0 \quad \forall g \in V_{-1} && \iff \quad \text{(see Lemma 6.7)} \\ &\textstyle\int_{-\infty}^{\infty} \hat{l}(\omega)\hat{\alpha}(\omega)\hat{\varphi}(\omega)\overline{\hat{c}(\omega)\hat{\varphi}(\omega)}d\omega = 0, \end{aligned}$$

for every periodic $\hat{l}$ with period $\frac{1}{2}$. All of $\hat{l}(\omega)$, $\hat{\lambda}(\omega)$, $\hat{c}(\omega)$ are periodic with period one, so

$$\begin{aligned} 0 &= \sum_{k=-\infty}^{\infty} \int_k^{k+1} \hat{l}(\omega)\hat{\alpha}(\omega)\hat{\varphi}(\omega)\overline{\hat{c}(\omega)\hat{\varphi}(\omega)}d\omega \\ &= \sum_{k=-\infty}^{\infty} \int_0^1 \underbrace{\hat{l}(\omega+k)\hat{\alpha}(\omega+k)\overline{\hat{c}(\omega+k)}}_{\text{periodic}}|\hat{\varphi}(\omega+k)|^2 d\omega \\ &= \sum_{k=-\infty}^{\infty} \int_0^1 \hat{l}(\omega)\hat{\alpha}(\omega)\overline{\hat{c}(\omega)}|\hat{(}\varphi(\omega+k)|^2 d\omega \quad \text{(use Theorem 1.5, page 43)} \\ &= \int_0^1 \hat{l}(\omega)\hat{\alpha}(\omega)\overline{\hat{c}(\omega)}d\omega. \end{aligned}$$

Since $\hat{l}(\omega)$ is periodic with period $\frac{1}{2}$, $\Longrightarrow$

$$\begin{aligned} 0 &= \int_0^{1/2} \hat{l}(\omega)\hat{\alpha}(\omega)\overline{\hat{c}(\omega)}d\omega + \int_{1/2}^1 \hat{l}(\omega)\hat{\alpha}(\omega)\overline{\hat{c}(\omega)}d\omega \quad (\omega = \nu + \frac{1}{2}) \\ &= \int_0^{1/2} \hat{l}(\nu)\hat{\alpha}(\nu + \frac{1}{2})\overline{\hat{c}(\nu + \frac{1}{2})}d\nu + \int_0^{1/2} \hat{l}(\omega)\hat{\alpha}(\omega)\overline{\hat{c}(\omega)}d\omega \\ &= \int_0^{1/2} \hat{l}(\omega)\left[\hat{\alpha}(\omega)\overline{\hat{c}(\omega)} + \hat{\alpha}(\omega + \frac{1}{2})\overline{\hat{c}(\omega + \frac{1}{2})}\right] d\omega. \end{aligned}$$

Here $\hat{l}(\omega)$ is *arbitrary*, so $[\ldots] = 0$, i.e.,

$$\hat{\alpha}(\omega)\overline{\hat{c}(\omega)} + \hat{\alpha}(\omega + \frac{1}{2})\overline{\hat{c}(\omega + \frac{1}{2})} = 0 \tag{11}$$

for (almost) all ω. Thus, $f \in W_{-1}$ if and only if (10) and (11) hold. If both $\hat{\alpha}(\omega) \neq 0$ and $\hat{\alpha}(\omega + \frac{1}{2}) \neq 0$, then (11) is equivalent to

$$\frac{\overline{\hat{c}(\omega + 1/2)}}{\hat{\alpha}(\omega)} = -\frac{\overline{\hat{c}(\omega)}}{\hat{\alpha}(\omega + 1/2)}$$

Denote this number by $\overline{\lambda(\omega + \frac{1}{2})}$. If only $\hat{\alpha}(\omega) \neq 0$, then we define

$$\lambda(\omega + \frac{1}{2}) = \frac{\hat{c}(\omega + \frac{1}{2})}{\overline{\hat{\alpha}(\omega)}},$$

and if only $\hat{\alpha}(\omega + \frac{1}{2}) \neq 0$, then we define

$$\lambda(\omega + \frac{1}{2}) = -\frac{\hat{c}(\omega)}{\overline{\hat{\alpha}(\omega + \frac{1}{2})}},$$

Note that we cannot have both $\hat{\alpha}(\omega) = 0$ and $\hat{\alpha}(\omega + \frac{1}{2}) = 0$ except in a set of measure zero since

$$\begin{cases} \hat{c}(\omega) &= -\lambda(\omega + \frac{1}{2})\overline{\hat{\alpha}(\omega + \frac{1}{2})} \quad \text{and} \\ \hat{c}(\omega + \frac{1}{2}) &= \lambda(\omega + \frac{1}{2})\overline{\hat{\alpha}(\omega)} \end{cases} \tag{12}$$

In both cases we end up with the formula $|\hat{\alpha}(\omega)|^2 + |\hat{\alpha}(\omega + \frac{1}{2})|^2 = 1$ (see formula (7) on page 53), which is valid for almost all ω. The function $\hat{\lambda}$ is periodic with period 1 since both $\hat{\alpha}$ and $\hat{c}$ are periodic. In addition, if we replace ω by $\omega + \frac{1}{2}$ in (12) and use the periodicity, then we get

$$\begin{aligned} \hat{c}(\omega + \frac{1}{2}) &= -\lambda(\omega)\overline{\hat{\alpha}(\omega)} \\ \hat{c}(\omega) &= \lambda(\omega)\overline{\hat{\alpha}(\omega + \frac{1}{2})}. \end{aligned}$$

Compare this to (12) $\implies \lambda(\omega) = -\lambda(\omega + \frac{1}{2})$. Thus, λ is anti-periodic with period $\frac{1}{2}$. By Lemma 6.7, page 64 and by (12), since $f \in V_0$, we have

$$\begin{aligned}
\|f\|_{L^2}^2 &= \int_0^1 |\hat{c}(\omega)|^2 d\omega \\
&= \int_0^1 |\lambda(\omega)|^2 |\hat{\alpha}(\omega + \frac{1}{2})|^2 d\omega \\
&= \left(\int_0^{1/2} + \int_{1/2}^1\right) |\lambda(\omega)|^2 |\hat{\alpha}(\omega + \frac{1}{2})|^2 d\omega \quad (|\lambda(\omega + \frac{1}{2})| = |\lambda(\omega)|) \\
&= \int_0^{1/2} |\lambda(\omega)|^2 (abs\hat{\alpha}(\omega)^2 + |\hat{\alpha}(\omega + \frac{1}{2})|^2) d\omega \quad \text{(see (7) on page 53)} \\
&= \int_0^{1/2} |\lambda(\omega)|^2. \quad \square
\end{aligned}$$

Corollary 6.9. *With the same assumptions as in Lemmas 6.7-6.8, we have*

iv) $f \in W_0 \iff \hat{f}(\omega) = \mu(\omega)\overline{\hat{\alpha}(\frac{w}{2} + \frac{1}{2})}\hat{\varphi}(\frac{\omega}{2})$, *where* $\mu(\omega + 1) = -\mu(\omega)$, *and*

$$\|f\|_{L^2}^2 = \int_0^1 |\mu(\omega)|^2 d\omega \ (< \infty)$$

Proof. We have $f \in W_0 \iff D_{1/2} \in W_{-1}$. By Lemma 6.8

$$\widehat{D_{1/2}f}(\omega) = \sqrt{2}\hat{f}(2\omega) = \lambda(\omega)\overline{\hat{\alpha}(\omega + 1/2)}\hat{\varphi}(\omega).$$

Replace ω by $\omega/2$, and put $\mu(\omega) = \frac{1}{\sqrt{2}}\lambda(\frac{\omega}{2})$. Then $\mu(\omega + 1) = -\mu(\omega)$, and by Lemma 6.8,

$$\begin{aligned}
\|f\|_{L^2}^2 &= \|D_{1/2}f\|_{L^2}^2 = \int_0^{1/2} |\lambda(\omega)|^2 d\omega \\
&\overset{\omega=\nu/2}{=} \frac{1}{2}\int_0^1 |\lambda(\nu/2)|^2 d\omega = \int_0^1 |\mu(\omega)|^2 d\omega
\end{aligned}$$

i.e.,

$$\hat{\alpha}(\omega)\overline{\hat{c}(\omega)} + \hat{\alpha}(\omega + 1/2)\overline{\hat{c}(\omega + 1/2)} = 0 \quad \text{(for (almost) all } w). \tag{13}$$

Thus $f \in W_{-1}$ if and only if (**??**) and (13) hold.

If $\hat{\alpha}(\omega + 1/2) \neq 0$, then we can define

$$\lambda(\omega + 1/2) = -\frac{\hat{c}(\omega)}{\overline{\hat{\alpha}(\omega + 1/2)}},$$

so that (13) gives in this case

$$\begin{cases} \hat{c}(\omega) = -\lambda(\omega + 1/2)\overline{\hat{\alpha}(\omega + 1/2)} \quad \text{and} \\ \hat{c}(\omega + 1/2) = \lambda(\omega + 1/2)\overline{\hat{\alpha}(\omega)}. \end{cases} \tag{14}$$

If, instead $\hat{\alpha}(\omega) = 0$ then we can define $\lambda(\omega + 1/2)$ by

$$\lambda(\omega + 1/2) = \frac{\hat{c}(\omega + 1/2)}{\overline{\hat{\alpha}(\omega)}},$$

and from (13) we again get (15). We know that at least one of $\hat{\alpha}(\omega)$ and $\hat{\alpha}(\omega + 1/2) \neq 0$, because $|\hat{\alpha}(\omega)|^2 + |\hat{\alpha}(\omega + 1/2)|^2 = 1$ (see formula (7) on page 53). Thus, (15) *always* holds. The function λ is periodic with period 1 since both $\hat{\alpha}$ and $\hat{c}$ are periodic. In addition, if we replace ω by $\omega + 1/2$ in (15) and use the periodicity, then we get

$$\begin{cases} \hat{c}(\omega + 1/2) = -\lambda(\omega)\overline{\hat{\alpha}(\omega)} \\ \hat{c}(\omega) = \lambda(\omega)\overline{\hat{\alpha}(\omega + 1/2)}. \end{cases} \tag{15}$$

Compare this to (15) $\implies$ $\lambda(\omega) = -\lambda(\omega + 1/2)$. Thus, λ is antiperiodic with period 1/2. □

On the other hand, part i) of Lemma 2.6 is also true if we replace V_0 by W_0 and φ by ψ, where ψ is an arbitrary mother wavelet (if such a mother wavelet exists):

Lemma 6.10. *Suppose that the MRA* $\{V_j\}_{j=-\infty}^{\infty}$ *has a mother wavelet* ψ. *Then*

$$f \in W_0 \iff \hat{f}(\omega) = \hat{d}(\omega)\hat{\psi}(\omega)$$

for some periodic $\hat{d}$ *with period* 1, *and*

$$\|f\|_{L^2}^2 = \int_0^1 |\hat{d}(\omega)|^2 d\omega.$$

Proof. Same as proof of part i) of Lemma 6.7, page 64. (Use Theorem 1.5 on page 43). □

Comparing this to Corollary 6.9 we realize how to construct a mother wavelet!

Theorem 6.11. *Let $\{V_j\}_{j=-\infty}^{\infty}$ be a MRA with scaling function φ. Then ψ* ***is a mother wavelet for this MRA*** $\iff$

$$\hat{\psi}(\omega) = \nu(\omega)\overline{\hat{\alpha}(\frac{\omega}{2} + \frac{1}{2})}\hat{\varphi}(\frac{\omega}{2}), \tag{16}$$

where $\nu(\omega + 1) = -\nu(\omega)$ and $|\nu(\omega)| = 1$ for almost all ω.

Proof. Necessity: Suppose that ψ is a mother wavelet. Then $\psi \in W_0$, so by Corollary 6.9, it has a representation

$$\hat{\psi}(\omega) = \nu(\omega)\overline{\hat{\alpha}(\frac{\omega}{2} + \frac{1}{2})}\hat{\varphi}(\omega), \quad \nu(\omega + 1) = -\nu(\omega).$$

In addition, by Theorem 1.4 on page 42, for almost all $\omega \in \mathbb{R}$,

$$\begin{aligned}
1 &= \sum_{k=-\infty}^{\infty} |\hat{\psi}(\omega+k)|^2 \\
&= \sum_{k=-\infty}^{\infty} \underbrace{|\hat{\nu}(\omega+k)|^2}_{|\hat{\nu}(\omega)|^2} \underbrace{|\hat{\alpha}(\frac{\omega}{2}+\frac{k+1}{2})}_{\substack{\text{periodic,}\\ \text{period } 2}}|^2 |\hat{\varphi}(\frac{\omega}{2}+\frac{k}{2})|^2 \\
&= |\hat{\nu}(\omega)|^2 \Big(\overset{\text{(even)}}{\sum_{k=2l}} + \overset{\text{(odd)}}{\sum_{k=2l+1}} \Big) |\hat{\alpha}(\frac{\omega}{2}+\frac{k+1}{2})|^2 |\hat{\varphi}(\frac{\omega}{2}+\frac{k}{2})|^2 \\
&= |\hat{\nu}(\omega)|^2 \Big[\sum_{l=-\infty}^{\infty} |\hat{\alpha}(l+\frac{\omega+1}{2})|^2 |\hat{\varphi}(\frac{\omega}{2}+l)|^2 \\
&\qquad + \sum_{l=-\infty}^{\infty} \underbrace{|\hat{\alpha}(l+\frac{\omega}{2}+1)}_{\substack{\text{periodic,}\\ \text{period } 1}}|^2 |\hat{\varphi}(\frac{\omega+1}{2}+l)|^2 \Big] \\
&= |\hat{\nu}(\omega)|^2 \Big[|\hat{\alpha}(\frac{\omega}{2}+\frac{1}{2})|^2 \sum_{l=-\infty}^{\infty} |\hat{\varphi}(\frac{\omega}{2}+l)|^2 \\
&\qquad + |\hat{\alpha}(\frac{\omega}{2})|^2 \sum_{l=-\infty}^{\infty} |\hat{\varphi}(\frac{\omega}{2}+\frac{1}{2}+l)|^2 \Big] \\
&\quad \text{(use Theorem 1.4, page 42)} \\
&= |\hat{\nu}(\omega)|^2 [|\hat{\alpha}(\frac{\omega}{2}+\frac{1}{2})|^2 + |\hat{\alpha}(\frac{\omega}{2})|^2] \\
&\quad \text{(use formula (7) on page 53)} \\
&= |\hat{\nu}(\omega)|^2.
\end{aligned}$$

Thus, $|\hat{\nu}(\omega)| = 1$ for almost all ω.

Sufficiency: Assume that (16) holds (page 70). The above computation shows that then $\{T_k\psi\}_{k=-\infty}^{\infty}$ is an orthogonal system of translates (see Theorem 1.4, page 42). By Corollary 6.9, $\psi \in W_0$. Since W_0 is invariant under integer translates (easy to prove), we have $T_k\psi \in W_0$ for all k, so the space spanned by $\{T_k\psi\}_{k=-\infty}^{\infty}$ (see Theorem **??** on page **??**) must be a subspace of W_0.

Is it all of W_0? Yes! Seen as follows.

Take any $g \in W_0$. Then by Corollary 6.9,

$$\hat{g}(\omega) = \mu(\omega)\overline{\hat{\alpha}(\omega/2 + 1/2)}\hat{\varphi}(\omega/2),$$

so we get from (16)

$$\hat{g}(\omega) = \frac{\hat{\mu}(\omega)}{\nu(\omega)}\hat{\psi}(\omega).$$

Here

$$\frac{\hat{\mu}(\omega+1)}{\nu(\omega+1)} = \frac{-\hat{\mu}(\omega)}{-\nu(\omega)} = \frac{\hat{\mu}(\omega)}{\nu(\omega)}$$

is periodic with period one. Moreover

$$\int_0^1 \left|\frac{\hat{\mu}(\omega)}{\nu(\omega)}\right|^2 d\omega < \infty.$$

Thus, by Theorem 1.5 on page 43, g belongs to the subspace spanned by $\{T_k\psi\}_{k=-\infty}^{\infty}$. □

Back to Theorem 6.11 on page 70: We can choose ν to be *any function* satifying $\nu(\omega+1) = -\nu(\omega)$ and $|\nu(\omega)| = 1$. The standard choice is to take

$$\nu(\omega) = -e^{-\pi i\omega}, \quad \text{(sometimes } \nu(\omega) = \pm e^{\pm\pi i\omega}\text{)}.$$

We shall do so in the sequel.

Definition 6.12. Let $\{V_j\}_{j=-\infty}^{\infty}$ be a MRA with scaling function φ. Then the *standard mother wavelet* associated with the MRA is the function ψ satifying

$$\hat{\psi}(\omega) = -e^{i\pi\omega}\overline{\hat{\alpha}(\omega/2 + 1/2)}\hat{\varphi}(\omega),$$

where α is the low-pass filter in Theorem 3.9 on page 53.

Theorem 6.13. *The standard mother wavelet defined above has the following expansion in terms of the scaling function φ: Let*

$$\beta(k) = (-1)^k\overline{\alpha(1-k)} \tag{17}$$

Then

$$\begin{aligned}
\psi(x) &= 2\sum_{k=-\infty}^{\infty}\beta(k)\varphi(2x-k), && (18)\\
\hat{\psi}(2\omega) &= \hat{\beta}(\omega)\hat{\varphi}(\omega), \quad \textit{where} && (19)\\
\hat{\beta}(\omega) &= -e^{-2\pi i\omega}\overline{\hat{\alpha}(\omega+1/2)}. && (20)
\end{aligned}$$

Moreover,

$$\sum_{k=-\infty}^{\infty} |\beta(k)|^2 = \frac{1}{2}, \quad \textit{and} \tag{21}$$

$$|\hat{\beta}(\omega)|^2 + |\hat{\beta}(\omega + \frac{1}{2})|^2 = 1, \quad \omega \in \mathbb{R}. \tag{22}$$

Proof. Homework. (Compare this to Theorem 3.9, on page 53). □

V.7 Examples of Wavelet Bases

V.7.1 The Haar Wavelet

It is easy to see that the Haar scaling function p on page 23 and the Haar wavelet on page 26 satisfy

$$p(x) = p(2x) + p(2x-1) \implies \alpha(0) = \alpha(1) = \frac{1}{2}$$
$$h(x) = p(2x) - p(2x-1) \implies \beta(0) = 1, \ \beta(1) = -1.$$

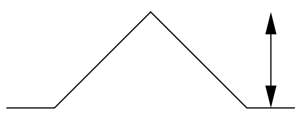
This agrees with formula (17) on page 72.

V.7.2 The Piecewise Linear Wavelet

By the formula on page 57,

$$\hat{\tilde{\varphi}}(\omega) = \frac{\hat{\varphi}(\omega)}{\sqrt{\frac{1}{3}(1 + 2\cos^2(\pi\omega))}},$$

where φ is the function



The transform of this function satisfies

$$\hat{\varphi}(\omega) = \cos^2(\frac{\pi\omega}{2})\hat{\varphi}(\frac{\omega}{2}).$$

Therefore,

$$
\begin{aligned}
\hat{\hat{\varphi}}(\omega) &= \frac{\cos^2(\frac{\pi\omega}{2})\sqrt{\frac{1}{3}(1+2\cos^2(\frac{\pi\omega}{2}))}}{\sqrt{\frac{1}{3}(1+2\cos^2(\pi\omega))}} \cdot \frac{\hat{\varphi}(\frac{\omega}{2})}{\sqrt{\frac{1}{3}(1+2\cos^2(\frac{\pi\omega}{2}))}} \\
&\quad \text{(use formula (5) on page 53)} \\
&= \hat{\alpha}(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2}).
\end{aligned}
$$

where

$$\hat{\alpha}(\omega) = \cos^2(\pi\omega)\sqrt{\frac{1+2\cos^2(\pi\omega)}{1+2\cos^2(2\pi\omega)}}.$$

By formula (17) on page 72,

$$
\begin{aligned}
\hat{\beta}(\omega) &= -e^{-2\pi i\omega}\overline{\hat{\alpha}(\omega+1/2)} \\
&= -e^{-2\pi i\omega}\cos^2(\pi\omega+\frac{\pi}{2})\sqrt{\frac{1+2\cos^2(\pi\omega+\frac{\pi}{2})}{1+2\cos^2(2\pi\omega+\pi)}} \\
&= -e^{-2\pi i\omega}\sin^2(\pi\omega)\sqrt{\frac{1+2\sin^2(\pi\omega)}{1+2\cos^2(2\pi\omega)}}
\end{aligned}
$$

By formula (19) on page 72,

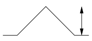
$$
\begin{aligned}
\hat{\psi}(\omega) &= \hat{\beta}(\frac{\omega}{2})\hat{\hat{\varphi}}(\frac{\omega}{2}) \\
&= -e^{-\pi i\omega}\sin^2(\frac{\pi\omega}{2})\sqrt{\frac{1+2\sin^2(\frac{\pi\omega}{2})}{1+2\cos^2(\pi\omega)}}\frac{\hat{\varphi}(\frac{\omega}{2})}{\sqrt{\frac{1}{3}(1+2\cos^2(\frac{\pi\omega}{2}))}} \\
&= \hat{\gamma}(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2}),
\end{aligned}
$$

where

$$\hat{\gamma}(\omega) = -\sqrt{3}e^{-2\pi i\omega}\sin^2(\pi\omega)\sqrt{\frac{1+2\sin^2(\pi\omega)}{[1+2\cos^2(2\pi\omega)][1+2\cos^2(\pi\omega)]}}.$$

From here we get

$$\psi(x) = 2\sum_{k=-\infty}^{\infty}\gamma(k)\varphi(2x-k),$$

where φ is the function  and γ is the inverse Fourier transform of $\hat{\gamma}$. See picture in Walnut on page 188. (Not quite correct: a shift is missing). (The coefficients $\gamma(k)$ are $\neq 0$ for all k, but they tend geometrically to zero as $k \to \pm\infty$).

V.7.3 The Shannon Wavelet

The Shannon wavelet is given by

$$\psi(x) = \frac{\sin(2\pi x) - \cos(\pi x)}{\pi(x - 1/2)} = \frac{\sin \pi(x - 1/2)}{\pi(x - 1/2)}(1 - 2\sin \pi x).$$

See picture in Walnut on page 189 (which appears to be correct).

V.7.4 The Strömberg-Meyer Wavelet

The Strömberg-Meyer Wavelet is constructed from the formulas on page 72, starting from the formula for $\hat{\varphi}$:

i) Use (5) on page 53 to get $\hat{\alpha}$

ii) Use (20) on page 72 to get $\hat{\beta}$

iii) Take ψ to be the inverse transform of

$$\hat{\psi}(\omega) = \hat{\beta}(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2}).$$

(The result depends on how you choose the original function $\hat{\varphi}$.

V.8 Spline Wavelets

Instead of using the triangular functions in V.7.2 on page **??** we can also use higher order "spline" functions: These are functions which are

- piecewise polynomial functions, which are
- patched together so that only the highest order derivative is discontinuous.
- Supported on a finite interval.

Definition 8.1. Let

$$B_0(x) = \begin{cases} 1, & -\frac{1}{2} \le x \le \frac{1}{2} \\ 0, & \text{otherwise,} \end{cases}$$

and for $n = 1, 2, 3, \ldots$, define

$$B_n(x) = (B_{n-1} * B_0)(x) = \int_{x-1/2}^{x+1/2} B_{n-1}(t)dt.$$

The function B_n is called the *B-spline* of order n. For $n \in \mathbb{Z}_+$, define

$$\tilde{B}_n(x) = T_{(n+1)/2}B_n(x) = B_n(x - \frac{n+1}{2}).$$

Lemma 8.2.

(a) $\tilde{B}_0$ *is the Haar scaling function*

(b) $\tilde{B}_1$ *is the function* φ *which we used in the construction of the piecewise linear wavelet*

(c)

$$\tilde{B}_2(x) = \begin{cases} \frac{1}{2}x^2, & x \in [0,1) \\ x^2 + 3x - \frac{3}{2}, & x \in [1,2) \\ \frac{1}{2}x^2 - 3x + \frac{9}{2}, & x \in [2,3] \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. Direct computation. □

Lemma 8.3.

(a) B_n *is supported in* $[-\frac{n+1}{2}, \frac{n+1}{2}]$, *and* $\tilde{B}_n$ *is supported in* $[0, n+1]$.

(b) B_n *and* $\tilde{B}_n$ *are* $n-1$ *times continuously differentiable for* $n \ge 1$.

(c) $\tilde{B}_n$ *is equal to a polynomial of degree* n *in each of the intervals* $[k, k+1]$, $0 \le k \le n$ *(and zero elsewhere).*

(d)

$$\begin{aligned}\hat{B}_n(\omega) &= \Big(\frac{\sin(\pi\omega)}{\pi\omega}\Big)^{n+1}, \quad \textit{and} \\ \hat{\tilde{B}}_n(\omega) &= e^{-\pi i(n+1)\omega}\Big(\frac{\sin(\pi\omega)}{\pi\omega}\Big)^{n+1}, \; n \geq 0.\end{aligned}$$

Proof. (a) - (b) are easy. (c) is proved by induction. (d) follows from Theorem 5.5 on page 17. ☐

As in the case of the piecewise linear multiresolution, the function $\tilde{B}_n$ does *not* define an orthonormal system of translates, but it does define a Riesz system of translates. The crusial function

$$M(\omega) = \sum_{k=-\infty}^{\infty} |\hat{g}(\omega + k)|^2$$

in Definition 2.1 on page 47 is now given by

$$\begin{aligned}M(\omega) &= \sum_{k=-\infty}^{\infty} \frac{|\sin\pi(\omega+k)|^{2n+2}}{|\pi(\omega+k)|^{2n+1}} \\ &\quad \text{(the numerator is periodic)} \\ &= \Big(\frac{\sin(\pi\omega)}{\pi}\Big)^{2n+2} \sum_{k=-\infty}^{\infty} \frac{1}{(\omega+k)^{2n+2}} \\ &= \Big(\frac{\sin(\pi\omega)}{\pi\omega}\Big)^{2n+2} + \Big(\frac{\sin(\pi\omega)}{\pi}\Big)^{2n+2} \sum_{k\neq 0} \frac{1}{(\omega+k)^{2n+2}}\end{aligned}$$

The function $M(\omega)$ is periodic with period 1, so to show that the frame condition in Definition 2.1 on page 47 holds it suffices to find $0 < A < B$ so that

$$A \leq M(\omega) \leq B \quad \text{for } \omega \in [-\frac{1}{2}, \frac{1}{2}].$$

The function $\frac{\sin x}{x}$ is a monotonely decreasing function in the interval $[0, \pi]$, so for $|\omega| \leq \frac{1}{2}$

$$\begin{aligned}M(\omega) &\geq \Big(\frac{\sin(\pi\omega)}{\pi\omega}\Big)^{2n+2} \geq \Big(\frac{\sin(\pi/2)}{\pi/2}\Big)^{2n+2} \\ &= \Big(\frac{2}{\pi}\Big)^{2n+2} > 0.\end{aligned}$$

Choose $A = \left(\frac{2}{\pi}\right)^{2n+2}$. On the other hand, for all ω in this interval $|\frac{\sin(\pi\omega)}{\pi\omega}| \leq 1$, $|\sin(\pi\omega)| \leq 1$, and for all $|k| \geq 0$,

$$|\omega + k| \geq |k| - \frac{1}{2},$$

so

$$M(\omega) \leq 1 + \frac{2}{\pi^{2n+2}} \sum_{k=1}^{\infty} \frac{1}{(k - \frac{1}{2})^{2n+2}} < \infty.$$

Choose B to be the right hand side. Then $0 < A \leq M(\omega) \leq B$, and we see that $\{T_k \tilde{B}_n\}$ is a Riesz system of translates.

We can now continue exactly as we did in Section V.7.2 in the piecewise linear case to get a family of wavelets consisting of m order splines.

V.9 Properties of the Scaling Functions

Througout this section we assume that

$$\begin{aligned} \varphi &\in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \quad \text{and} \\ \psi &\in L^1(\mathbb{R}) \cap L^2(\mathbb{R}). \end{aligned}$$

In particular,

$$\begin{aligned} \int_{-\infty}^{\infty} |\varphi(x)| dx &< \infty \quad \text{and} \\ \int_{-\infty}^{\infty} |\psi(x)| dx &< \infty. \end{aligned}$$

In particular, this means we *exclude* the shannon wavelet (which has $\int_{-\infty}^{\infty} |\varphi(x)| dx = \infty$).

Note. Much of what is said below is true under *weaker* assumptions on φ and ψ. The important thing is to assume that $\hat{\varphi}$ is continuous at zero.

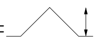
Theorem 9.1. *Let $\{V_j\}_{j=-\infty}^{\infty}$ be a MRA with scaling function φ. If $\varphi \in L^1(\mathbb{R})$, then*

$$|\hat{\varphi}(0)| = 1.$$

Proof (outline). Choose an arbitrary $\hat{f} \in L^2(\mathbb{R})$, so that

A) $\hat{f}(\omega) = 0$ for $|\omega| \geq 1$ (for example)

B) $\|\hat{f}\|_{L^2(\mathbb{R})} \neq 0$.

For example $\hat{f}(\omega) =$ . Let f be the inverse Fourier transform of $\hat{f}$. We project f onto V_j using P_j:

$$f_j = P_j f = \sum_{k=-\infty}^{\infty} \langle f, \varphi_{j,k} \rangle \varphi_{j,k}.$$

Since $\hat{\varphi}_{j,k}(\omega) = 2^{-j/2}\hat{\varphi}(2^{-j}\omega)e^{-2\pi i 2^{-j}k\omega}$, and since $\langle f, \varphi_{j,k} \rangle = \langle \hat{f}, \hat{\varphi}_{j,k} \rangle$ (Parseval), we get from Bessel's equality ((iii) on page 12)

$$\begin{aligned}
\|f_j\|_{L^2}^2 &= \sum_{k=-\infty}^{\infty} |\langle f, \varphi_{j,k} \rangle|^2 \\
&= \sum_{k=-\infty}^{\infty} \left| \int_{\mathbb{R}} \hat{f}(\omega) 2^{-j/2} \overline{\hat{\varphi}(2^{-j}\omega)} e^{-2\pi i 2^{-j}k\omega} d\omega \right|^2 \\
&= \sum_{k=-\infty}^{\infty} \left| \int_{-1}^{1} \hat{f}(\omega) 2^{-j/2} \overline{\hat{\varphi}(2^{-j}\omega)} e^{-2\pi i 2^{-j}k\omega} d\omega \right|^2
\end{aligned}$$

(since $\hat{f}(\omega) = 0$ for $|\omega| \geq 1$). The sequence $\{2^{j/2}e^{-2\pi i 2^{-j}k\omega}\}$ is an orthogonal basis for $L^2((-2^{j-1}, 2^{j-1}))$. Choose j so large that $2^{j-1} > 1$, and use Bessel's equality once more (in the opposite direction) to get

$$\|f_j\|_{L^2}^2 = \int_{-1}^{1} |\hat{f}(\omega)|^2 |\hat{\varphi}(2^{-j}\omega)|^2 d\omega.$$

Now let $j \to \infty$. Then $\hat{\varphi}(2^j\omega) \to \hat{\varphi}(0)$ uniformly on $[-1, 1]$, so we get

$$\begin{aligned}
\lim_{j\to\infty} \|f_j\|_{L^2}^2 &= |\hat{\varphi}(0)|^2 \int_{-1}^{1} |\hat{f}(\omega)|^2 d\omega \\
&= |\hat{\varphi}(0)|^2 \|\hat{f}\|_{L^2}^2.
\end{aligned}$$

On the other hand, by using condition (c) in Definition 3.2 on page 50 (in the form of Lemma 3.6(c), page 52) we get

$$\lim_{j\to\infty} \|f_j\|_{L^2}^2 = \|f\|_{L^2}^2 = \|\hat{f}\|_{L^2}^2.$$

Substitute above $\implies (1 - |\hat{\varphi}(0)|^2)\|\hat{f}\|_{L^2}^2 = 0 \implies |\hat{\varphi}(0)| = 1.$ $\square$

Corollary 9.2. *Assuming still that both* $\varphi \in L^1(\mathbb{R})$ *and* $\psi \in L^1(\mathbb{R})$ *we have*

$$\int_{-\infty}^{\infty} \psi(x)dx = 0 \quad (\iff \hat{\varphi}(0) = 0).$$

Proof. $\hat{\varphi}$ and $\hat{\psi}$ are continuous since $\varphi \in L^1(\mathbb{R})$ and $\psi \in L^1(\mathbb{R})$ (this is part of the Riemann-Lebesgue Lemma). By formula (5) on page 53,

$$\hat{\alpha}(\omega) = \frac{\hat{\varphi}(2\omega)}{\hat{\varphi}(\omega)} \quad \text{(whenever } \hat{\varphi}(\omega) \neq 0\text{)}.$$

This together with Theorem 9.1 on page 78 implies that $\hat{\alpha}(0) = 1$, and that $\hat{\varphi}$ is continuous in a neighborhood around zero. This plus formula (7) on page 53 gives $\hat{\alpha}(\frac{1}{2}) = 0$, Plug this into the formula (20) on page 72 to get $\hat{\psi}(0) = 0$. But

$$\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(x)dx. \quad \square$$

Corollary 9.3. *If* $\varphi \in L^1(\mathbb{R})$, *then*

$$\hat{\varphi}(n) = 0 \quad \textit{for } n = \pm 1, \pm 2, \ldots (n \neq 0).$$

Proof. Follows from Theorem 9.1 on page 78 and the formula

$$\sum_{n=-\infty}^{\infty} |\hat{\varphi}(\omega + n)|^2 = 1$$

(this is true now for *all* ω and not just *almost* all ω, since $\varphi \in L^1(\mathbb{R})$). $\square$

Corollary 9.4. *If* $\varphi \in L^1(\mathbb{R})$, *then* φ *is a "partition of the unity" in the sense that*

$$\sum_{n=-\infty}^{\infty} \varphi(x+n) = 1 \quad \textit{for (almost) all } x.$$

Proof. The function $\sum_{n=-\infty}^{\infty} \varphi(x+n)$ is integrable over $[0, 1)$:

$$\begin{aligned}
\int_0^1 \Big| \sum_{n=-\infty}^{\infty} \varphi(x+n) \Big| dx &\leq \int_0^1 \sum_{n=-\infty}^{\infty} |\varphi(x+n)| dx \\
&= \sum_{n=-\infty}^{\infty} \int_0^1 |\varphi(x+n)|^2 dx \\
&= \int_0^1 |\varphi(x)| dx < \infty.
\end{aligned}$$

It is periodic (easy to see) with period one. Therefore, for all integers k, if we compute the Fourier Series, we get

$$\begin{aligned}
\int_0^1 e^{-2\pi ik\omega} \sum_{n=-\infty}^{\infty} \varphi(x+n)dx &= \sum_{n=-\infty}^{\infty} \int_0^1 e^{-2\pi ik\omega}\varphi(x+n)dx \\
&\overset{x+n=y}{=} \sum_{n=-\infty}^{\infty} \int_n^{n+1} e^{-2\pi ik\omega}\varphi(x)dx \\
&= \hat{\varphi}(k) = \begin{cases} 1, & k=0, \\ 0, & k \neq 0. \end{cases}
\end{aligned}$$

The only function which has the above Fourier series is the function $\equiv 1$. □

Comment 9.5. A converse of Theorem 9.1 on page 78 is also true: If $\hat{\varphi}(0) = 1$, then properties (d) and (e) in Definition 3.2 implies property (c) (recall that our proof of Theorem 9.1 used property (c) in a crucial way). Thus, if the other properties in Definition 3.2 hold, then

$$(c) \iff |\hat{\varphi}(0)| = 1$$

(if $\varphi \in L^1(\mathbb{R})$). In many cases it is much easier to check that $|\hat{\varphi}(0)| = 1$ then to check (c) directly. See Gripenberg's note for the proof.

Comment 9.6. Usually one standardizes $\hat{\varphi}(0)$ to be

$$\hat{\varphi}(0) = 1$$

(simply replace φ by $\frac{\varphi}{\hat{\varphi}(0)}$).