Chapter IV

The Discrete Haar Transform

IV.1 Motivation

According to the discussion on page 31, after we have projected $L^2(\mathbb{R})$ onto V_N, we are dealing with piecewise constant functions. Since the functions are constant we may as well represent them by changing the "continuous time variable v" into a *discrete time variable* k.

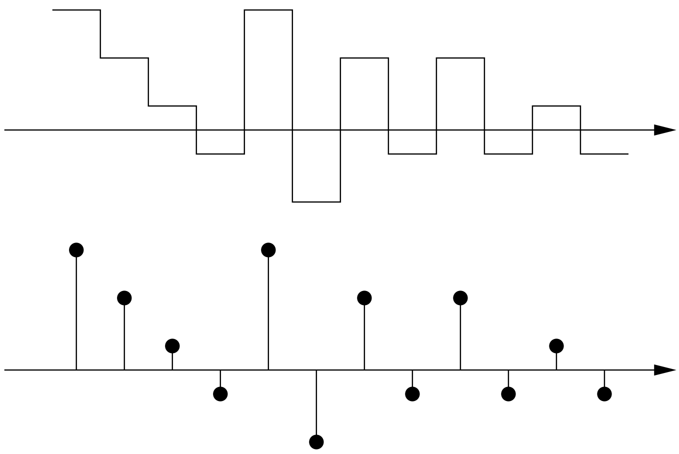

Figure IV.1: Function $f \in V_N$ replaced by a sequence

Every time we pass from one scale to the next courser scale the **distance between two points doubles**.

There is a one-to-one correspondence: **Given** $f \in V_n$ **we get the sequence** $\langle f, h_{N,k} \rangle$**. Given the sequence** $\langle f, h_{N,k} \rangle$ **we can recreate**

$f = \sum_k \langle f, h_{N,k} \rangle h_{N,k}$.

Recall: One step in the "analysis" is to write $f \in V_N$ in the form

$$f = f_{N-1} + g_{N-1} = \underbrace{P_{N-1}f}_{\substack{\text{courser} \\ \text{average}}} + \underbrace{Q_{N-1}f}_{\substack{\text{detail} \\ \text{information}}}$$

Means what? Since $f \in V_N$, we have

$$f = P_N f = \sum_{k=-\infty}^{\infty} c_0(k) p_{N,k}$$

$$P_{N-1} f = \sum_{k=-\infty}^{\infty} c_1(k) p_{N-1,k}$$

where $c_j(k) = \langle f, p_{N-j,k} \rangle$, $j = 1, 2$.

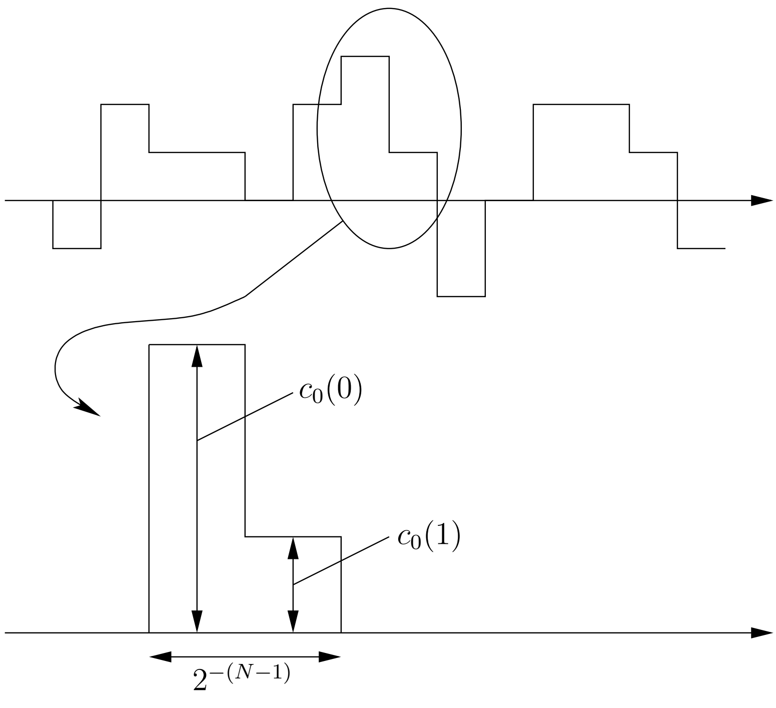

Figure IV.2:

Look at the composition on page 26: We get $c_1(0)$ by computing

$$\text{Put:} \quad c_j(k) = \langle f, p_{N-j,k} \rangle$$
$$d_j(k) = \langle f, h_{N-j,k} \rangle$$

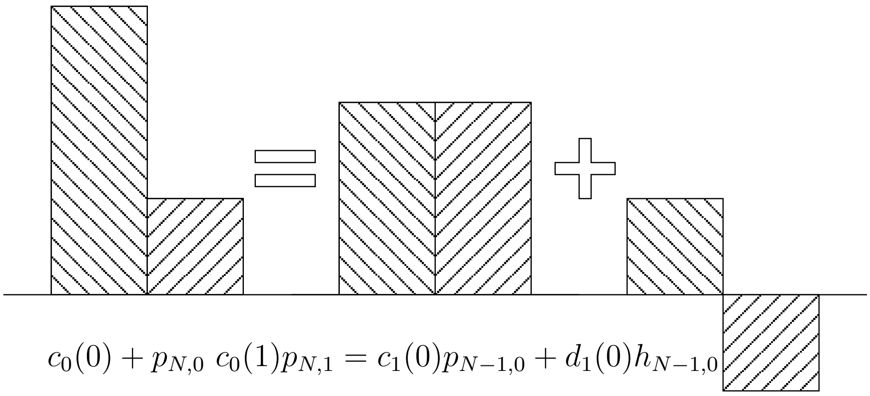

Figure IV.3:

From this it is in principle very simple to solve for the coefficients $c_1(0)$ and $d_1(0)$ in terms of $c_0(0)$ and $c_0(1)$. The only problem is the scaling factors $2^{j/2}$ in the definitions of $p_{j,k}$ and $h_{j,k}$. Anyway, the answer is:

Lemma 1.1. *Expand $f \in V_N$ into*

$$\underbrace{\sum_{k=-\infty}^{\infty} c_0(k)p_{N,k}}_{f} = \underbrace{\sum_{k=-\infty}^{\infty} c_1(k)p_{N-1,k}}_{P_{N-1}f} + \underbrace{\sum_{k=-\infty}^{\infty} d_1(k)h_{N-1,k}}_{Q_{N-1}f}$$

where

$$\left.\begin{aligned} c_0(k) &= \langle f, p_{N,k}\rangle \\ c_1(k) &= \langle f, p_{N-1,k}\rangle \end{aligned}\right\} \quad \text{“averages”}$$

$$d_1(k) = \langle f, h_{N-1,k}\rangle \qquad \text{“differences”}$$

Then

$$\begin{aligned} {\scriptstyle\text{courser averages}}\} \quad c_1(k) &= \frac{1}{\sqrt{2}}[c_0(2k)+c_0(2k+1)] \\ {\scriptstyle\text{corresponding difference}}\} \quad d_1(k) &= \frac{1}{\sqrt{2}}\underbrace{[c_0(2k)-c_0(2k+1)]}_{\text{averages, scale } 2^{-N}}. \end{aligned}$$

Proof. Homework. □

This means that in the analysis part we can **forget about the functions $p_{j,k}$ and $h_{j,k}$ and only work with the sequences $c_j(k)$ and $d_j(k)$.** However, the functions $p_{j,k}$ and $h_{j,k}$ are still needed in two steps:

- The initial projection onto V_N.
- The final stage when we "synthesize" back the original function from the coefficients.

IV.2 The Discrete Haar Transform (DHT)

We assume that we have a given sequence $\{c_N(k)\}_{k=-\infty}^{\infty}$ of "averages of order N", where N is fixed.

Definition 2.1. The *one step Discrete Haar Transform* maps the sequence

$$\{c_j(k)\}_{k=-\infty}^{\infty}$$

into the two sequences

$$\{c_{j+1}(k)\}_{k=-\infty}^{\infty} \quad \text{and} \quad \{d_{j+1}(k)\}_{k=-\infty}^{\infty}$$

given by

$$\begin{cases} c_{j+1}(k) &= \frac{1}{\sqrt{2}}[c_j(2k) + c_j(2k+1)], \\ d_{j+1}(k) &= \frac{1}{\sqrt{2}}[c_j(2k) - c_j(2k+1)]. \end{cases}$$

In matrix form this become:

$$\begin{bmatrix} c_{j+1}(k) \\ d_{j+1}(k) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_j(2k) \\ c_j(2k+1) \end{bmatrix}$$

Lemma 2.2. *The one step DHT is invertible and the original sequence $\{c_j(k)\}_{k=-\infty}^{\infty}$ can be reconstructed from the two sequences $\{c_{j+1}(k)\}_{k=-\infty}^{\infty}$ and $\{d_{j+1}(k)\}_{k=-\infty}^{\infty}$ by*

$$\begin{bmatrix} c_j(2k) \\ c_j(2k+1) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_{j+1}(k) \\ d_{j+1}(k) \end{bmatrix}.$$

Proof. Easy. □

Note. This is the "same" transformation as the original one. It "inverts itself".

Note 2.3. Often the original sequence $c_0(k)$ is of finite length. The next pair of sequences $c_1(k)$ and $d_1(k)$ are then only *half as long.* If we repeat the same step with $c_0(k)$ replaced by $c_1(k)$, then the new sequences that we get are again only half as long. It is therefore convenient to take **length = power of** 2 (put extra zeros at end, if necessary).

In the *multistep* version of DHT we leave $d_1(k)$ as it is. However, $c_1(k)$ is another sequence of "averages", so it can apply the "same transform" again. And so on.

Definition 2.4. Given $J \in \mathbb{N}$ (number of steps) and N (length of original sequence is 2^N) and a sequence $\{c_0(k)\}_{k=0}^{2^N-1}$ we define the DHT of $\{c_0(k)\}_{k=0}^{2^N-1}$ to be the sequences

$$\Big\{d_j(k)\Big\}_{\substack{1\le j\le J\\ 0\le k\le 2^{N-j}-1}} \quad \text{and} \quad \Big\{c_j(k)\Big\}_{0\le k\le 2^{N-J}-1} \tag{1}$$

recursively by

$$\begin{bmatrix} c_{j+1}(k) \\ d_{j+1}(k) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_j(2k) \\ c_j(2k+1) \end{bmatrix}, \quad 0 \le j \le J-1.$$

Lemma 2.5. *The multistep DHT can be inverted: Given the sequnce* (1) *above we can recover the original sequence* $\{c_0(k)\}_{k=0}^{2^N-1}$ *recursively from the formula*

$$\begin{bmatrix} c_j(2k) \\ c_j(2k+1) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_{j+1}(k) \\ d_{j+1}(k) \end{bmatrix}.$$

To get back the *original* continuous time interpretation, we only have to add two more steps:

- The projection into the original space V_N (which gives us the original sequence c_0). This is the *first analysis* step.

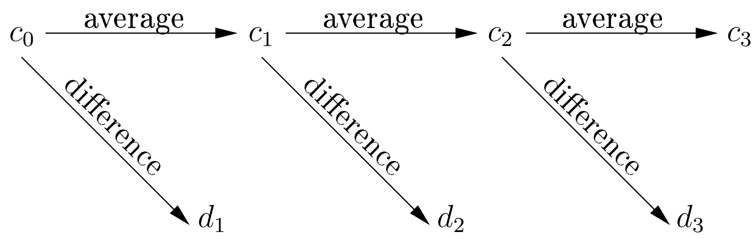

Figure IV.4: Analysis

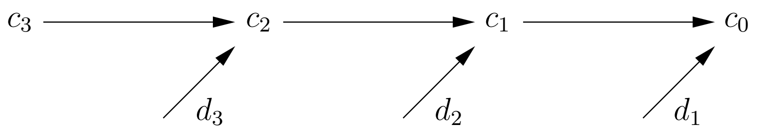

Figure IV.5: Synthesis

- The convertion of the coefficients c_0 back into a function. This is the *last* step in the synthesis procedure.

In these two steps we **need only the scaling functions** $p_{N,k}$ but **the wavelets** $h_{j,k}$ **are never needed!** (Analysis and final synthesis takes place in the space V_N).