Chapter III

The Haar System

III.1 Dyadic Step Functions and the Haar Scaling Function

Definition 1.1. A *dyadic* interval is an interval of the type

$$I_{j,k} = [2^{-j}k, 2^{-j}(k+1)).$$

A *dyadic step function* with scale j is a function which is constant on each interval $I_{j,k}$ (with j fixed).

Definition 1.2. The *Haar scaling functions* of order j are given by:

$$p_{j,k}(x) = 2^{j/2}p(2^j x - k),$$

where

$$p(x) = \begin{cases} 1, & 0 \leq x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 1.3. *We have the alternative formula*

$$p_{j,k}(x) = \begin{cases} 2^{j/2}, & x \in I_{j,k} \\ 0, & \textit{elsewhere.} \end{cases}$$

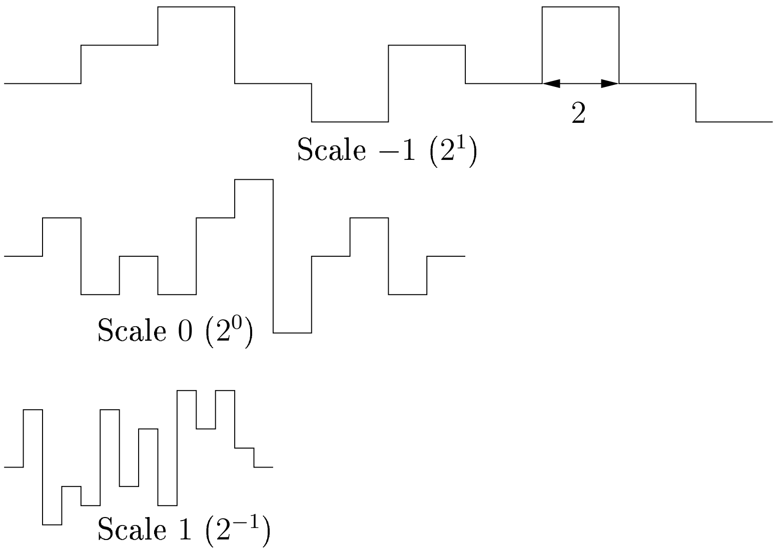

Figure III.1: Dyadic step functions

Proof.

$$\begin{aligned} p_{j,k}(x) \neq 0 &\iff 0 \leq 2^j x - k < 1 \\ &\iff k \leq 2^j x < k+1 \\ &\iff x \in [2^{-j}k, 2^{-j}(k+1)). \quad \square \end{aligned}$$

Notation 1.4. $V_j = \{$ set of all dyadic step functions of scale j which belong to $L^2(\mathbb{R})\}$. We call V_j the Haar *approximation space* of order j.

Thus: V_j consists of those functions of the type drawn in Figure III.1 on page 24 which satisfy $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$.

Lemma 1.5.

i) $V_j \subset V_{j+1}$ *for all* j,

ii) $\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$,

iii) $\bigcup_{j=-\infty}^{\infty} V_j$ *is* dense *in* $L^2(\mathbb{R})$.

Proof.

i) Obvious.

ii) If $f \in \bigcap_{j=-\infty}^{\infty} V_j$, then f is a constant, and the only constant which is in L^2 is zero.

iii) This says that given *any* $f \in L^2(\mathbb{R})$, we can find some $f_j \in V_j$ (for some sufficiently large j). The proof of this requires some extra knowledge of L^2:

Step 1: There is a function h which is *continuous* and has compact support so that

$$\|f - h\|^2 = \int_{-\infty}^{\infty} |h(x) - f(x)|^2 dx < \frac{\varepsilon}{2}.$$

Step 2: There is a dyadic step function with compact support f_j such that

$$\|f_j - h\|^2 \leq \frac{\varepsilon}{2}.$$

(We skip the details.) □

Theorem 1.6. *Fix any $j \in \mathbb{Z}$. Then the set of functions*

$$\{p_{j,k}\}_{k=-\infty}^{\infty}$$

is an orthonormal basis for V_j.

Proof. Homework. □

III.2 The Haar (Wavelet) System

Question 2.1. Since $V_j \subset V_{j+1}$, there are some functions in V_{j+1} which are orthogonal to all functions in V_j. What do they look like? In other words: Try to find the space W_j = orthogonal complement to V_j in V_{j+1}.

Notation 2.2. $V_{j+1} = V_j \oplus W_j$.

Solution: Each $f \in W_j$ belongs to V_{j+1} and is orthogonal to every p_j (since $p_j \in V_j$).

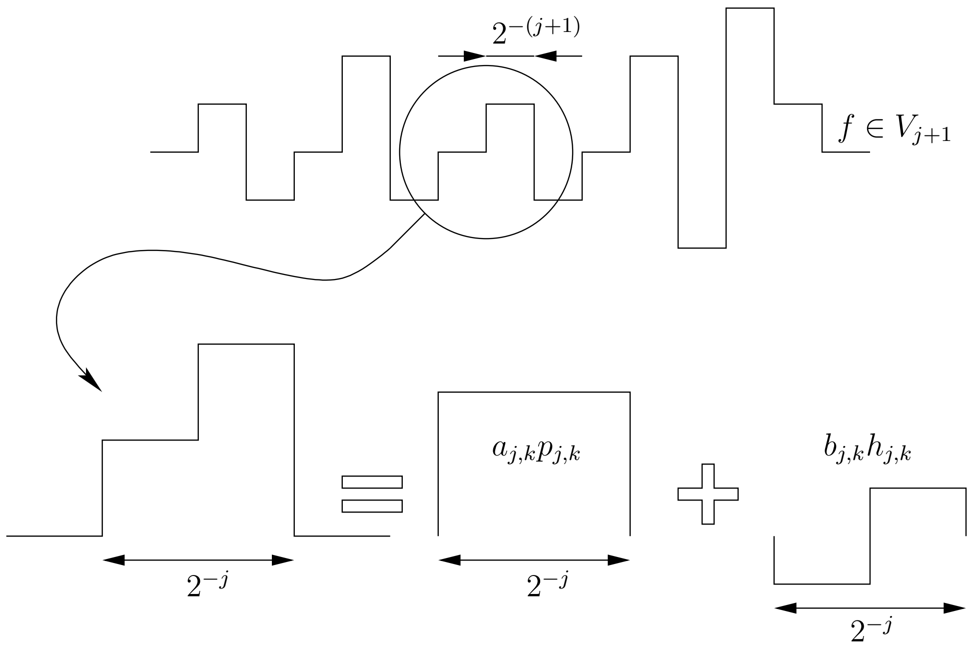

Figure III.2: The Haar system (of wavelets)

The orthogonality condition $p_{j,k} \perp f$ says:

$$\begin{aligned} \langle p_{j,k}, f \rangle = 0 \quad &\iff \quad \int_{k2^{-j}}^{(k+1)2^{-j}} f(x)dx = 0 \\ &\iff \quad \text{The average of } f \text{ over any dyadic interval } I_{j,k} \text{ is zero} \\ &\iff \quad f \text{ can be written as a sum of functions } h_{j,k} \text{ of the} \\ & \quad \text{type in Figure III.2} \end{aligned}$$

These functions have a name:

Definition 2.3. The *Haar system* (of wavelets) is the family

$$h_{j,k}(x) = 2^{j/2}h(2^j x - k), \quad j,k \in \mathbb{Z}$$

where h is the Haar wavelet:

$$h(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2}, \\ -1, & \frac{1}{2} \le x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.4. *Fix any $j \in \mathbb{Z}$. Then the set of functions (of order j)*

$$\{h_{j,k}\}_{k=-\infty}^{\infty}$$

is an orthonormal basis for W_j.

Proof (outline). Since $p_{j/2,k}$ is an orthonormal basis of V_{j+1} and $W_j \subset V_{j+1}$, every function in W_j has an expansion of the type

$$f(x) = \sum_{k=-\infty}^{\infty} c_k p_{j/2,k}(x).$$

In dyadic interval of length 2^{-j} the average of f is zero (since $f \in W_j$), so by combining two of the functions $p_{j/2,k}$ we get a multiple of $h_{j,k}$ (see last page). This leads to a sum of the type

$$f(x) = \sum_{k=-\infty}^{\infty} d_k h_{j,k}.$$

Conversely, every sum of this type is orthogonal to every function in V_j since

$$p_{j,k}(x) \perp h_{j,k} \quad \forall j, k$$

(easy to check). □

Theorem 2.5 (Splitting Theorem)**.**

i) *$V_{j+1} = V_j \oplus W_j$ (i.e., $V_j \perp W_j$ and every $f \in V_{j+1}$ can be written as $g + r$ where $g \in V_j$ and $r \in W_j$).*

ii) *Every $f \in V_{j+1}$ has an expansion*

$$f = \sum_{k=-\infty}^{\infty} a_{j,k} p_{j,k} + \sum_{k=-\infty}^{\infty} b_{j,k} h_{j,k},$$

with $p_{j,k}$ and $h_{j,k}$ as in Definitions 1.2 and 2.3.

iii) *The set of functions*

$$\{p_{j,k}\}_{k=-\infty}^{\infty} \bigcup \{h_{j,k}\}_{k=-\infty}^{\infty}$$

is an orthonormal basis in V_{j+1}.

Proof.

i) This is how W_j was defined, i.e., W_j was defined to be the orthogonal complement in V_{j+1} to V_j. See pages 25 to 27.

ii) Follows from i) and Theorems 1.6 and 2.4.

iii) Follows from ii) and part iv) of Theorem 2.3, page 12. Note that $h_{j,k} \perp p_{j,l}$ for all k, l since $p_{j,l} \in V_j$ and $h_{j,k} \in W_j$, and $V_j \perp W_j$. □

Lemma 2.6. *$W_j \perp W_l$ for $j \neq l$.*

Proof. If $l < j$, then $W_j \perp V_j$, and $W_l \subset V_{l+1} \subset V_j$. □

By *repeating* this splitting over and over again we get the following result:

$$\begin{aligned} V_j &= W_{j-1} \oplus V_{j-1} \\ &= W_{j-1} \oplus W_{j-2} \oplus V_{j-2} \\ &= W_{j-1} \oplus W_{j-2} \oplus W_{j-3} \oplus V_{j-3} \\ &= \ldots \end{aligned}$$

This leads to

Theorem 2.7. *For every $j, J \in \mathbb{Z}$, with $j > J$, the set of functions*

$$\underbrace{\{p_{J,k}\}_{k=-\infty}^{\infty}}_{\text{"averages" of order } J} \bigcup \underbrace{\{h_{l,k}\}_{\substack{-\infty<k<\infty \\ J \le l < j}}}_{\text{"differences" of order } k}$$

is an orthonormal basis in V_j.

Proof. We have

$$V_j = W_{j-1} \oplus W_{j-2} \oplus \ldots \oplus W_J \oplus V_J,$$

and $\{p_{J,k}\}_{k=-\infty}^{\infty}$ is a basis for V_J, and $\{h_{l,k}\}_{k=-\infty}^{\infty}$ is a basis for W_l. □

Here we let $j \to \infty$. The set V_j increases with j, and it tends to all of $L^2(\mathbb{R})$ as $j \to \infty$ (because of property iii) in Lemma 1.5 on page 24). Therefore:

Theorem 2.8. *For every $J \in \mathbb{Z}$, the set of functions*

$$\underbrace{\{p_{J,k}\}_{k=-\infty}^{\infty}}_{\text{"averages"}} \bigcup \underbrace{\{h_{j,k}\}_{\substack{j\geq J \\ -\infty<k<\infty}}}_{\text{"differences"}}$$

is an orthonormal basis in $L^2(\mathbb{R})$.

Proof. Easy to see that this sequence is orthonormal. If f is orthogonal to all of these functions, then by Theorem 2.7, f is orthogonal to V_j. This is true for all $j \geq J$. By Lemma 1.5, $\bigcup_{j=J}^{\infty} V_j$ is dense in $L^2(\mathbb{R})$, and therefore $f = 0$. Thus this is a basis. □

We can also let $J \to -\infty$, and just keep going in the expression on the top of this page. This leads to:

Theorem 2.9. *The set of functions*

$$\{h_{j,k}\}_{j,k=-\infty}^{\infty}$$

is an orthonormal basis in $L^2(\mathbb{R})$.

Proof. Easy to see that it is orthonormal. To see that it is a *basis* we can e.g. argue as follows: By Theorem 2.8, every $f \in L^2(\mathbb{R})$ has an expansion (for each fixed J)

$$f(x) = \sum_{k=-\infty}^{\infty} a_{J,k} p_{J,k}(x) + \sum_{k=-\infty}^{\infty} \sum_{j=J}^{\infty} b_{j,k} h_{j,k}(x),$$

where

$$\begin{aligned} a_{j,k} &= \langle f, p_{j,k} \rangle \\ b_{j,k} &= \langle f, h_{j,k} \rangle \end{aligned}$$

If $f \perp h_{j,k}$ for all $j \geq J$ and $k \in \mathbb{Z}$ then $b_{j,k} = 0$ for all j and k, so by Theorem 1.6, $f \in V_J$. This is true for all $J \in \mathbb{Z}$, so

$$f \in \bigcap_{J=-\infty}^{\infty} V_J.$$

By property ii) in Lemma 1.5, page 24, $f = 0$. Thus, only the zero function is orthogonal to all $h_{j,k}$, so $\{h_{j,k}\}$ is a *basis*. □

Comment 2.10. All of Theorems 2.7 - 2.9 are "important" for different reasons:

- Theorem 2.9 is important in the mathematical *theory*.
- Theorem 2.8 is more "practical": Instead of using arbitrary "course" scales we can "stop" at any time we please by adding the "average" functions $p_{j,k}$ to the basis.
- Theorem 2.7 is the one which is actually used in wavelet expansions. We first "project" an arbitrary $f \in L^2(\mathbb{R})$ onto the "fine" approximation space V_j and then do a "wavelet decomposition" using "differences" and some final "averages" on the scale J.

The Haar wavelets have a very nice "localization" property:

Theorem 2.11. *Theorems 2.7 - 2.8 remains true if we replace $L^2(\mathbb{R})$ by $L^2(0,1)$ (= L^2-functions defined on $(0,1)$) with the following modifications:*

A) Throughout we take $J \geq 0$ and $j \geq 0$.

B) We only include those values of k where $h_{j,k}(x) = 0$ for $x \notin [0,1)$.

Proof. True because each of the basis functions is either $= 0$ for all $x \in [0,1)$ or $= 0$ for all $x \notin [0,1)$ (as long as $j \geq 0$ and $J \geq 0$). □

III.3 The Haar Approximation Operator

By Lemma 1.5 iii), every function $f \in L^2(\mathbb{R})$ can be approximated, within an arbitrary tolerance ε, by a function in some of the space V_j (the smaller the ε, the bigger we have to choose j).

By Theorem 3.7 on page 14, there is always a *best* approximation $f_j \in V_j$ to V_j (the one that minimizes $\|f - f_j\|$), and by Theorem 3.7 and Theorem 1.6, the best approximation is given by

$$f_j = P_j f = \sum_{k=-\infty}^{\infty} \langle f, p_{j,k} \rangle p_{j,k} \tag{1}$$

(where $p_{j,k}$ are the Haar scaling functions of order j)

Definition 3.1. We call the operator P_j defined in (1) the *approximation operator* of order j (induced by the Haar system).

Theorem 3.2. *The approximation operators P_j has the following properties:*

i) *P_j is the orthogonal projection of $L^2(\mathbb{R})$ onto the approximation space V_j*

ii) *$P_j f = f$ whenever $f \in V_J$ for some $J \leq j$*

iii) $\lim_{j\to\infty} P_j f = f$ *for all* $f \in L^2(\mathbb{R})$

iv) $\lim_{j\to-\infty} P_j f = 0$ *for all* $f \in L^2(\mathbb{R})$

v) *$P_j P_J f = P_J P_j f = P_J f$ whenever $J \leq j$.*

Proof.

i) This follows from Theorem 3.7 (page 14) and Theorem 1.6 (page 25).

ii) If $f \in V_J$ then $f \in V_j$ (we have $V_J \subset V_j$ for $J \leq j$). We know that for all $f \in V_j$ we have

$$f = \sum_{k=-\infty}^{\infty} \langle f, p_{j,k} \rangle p_{j,k}$$

(see Theorem 1.6, page 25, and Theorem 2.3, page 12). The right-hand side is $P_j f$. Thus, $P_j f = f$ if $f \in V_J$ for some $J \leq j$.

iii) By Lemma 1.5 (page 24), there is a sequence $g_j \in V_j$ so that

$$\|g_j - f\| \to 0 \quad \text{as } j \to \infty.$$

The function $f_j = P_j f$ is the *best* approximation to f in V_j, so $\|f_j - f\| \leq \|g_j - f\|$. Therefore also $\|f_j - f\| \to 0$ as $j \to \infty$. This is the same thing as

$$\lim_{j\to\infty} f_j = f.$$

iv) By Theorem 2.9 on page 29, for each J,

$$f = P_J f + \sum_{k=-\infty}^{\infty} \sum_{j=J}^{\infty} \langle f, h_{j,k} \rangle h_{j,k}$$

so

$$P_J f = f - \sum_{k=-\infty}^{\infty} \sum_{j=J}^{\infty} \langle f, h_{j,k} \rangle h_{j,k}.$$

By Theorem 2.9 (page 29) and Theorem 2.3 iv) (page 12) the right hand side tends to zero as $J \to -\infty$. Thus $P_J f \to 0$ as $J \to -\infty$.

v) Since $P_J f \in V_J$ it follows from ii) that $P_j P_J f = P_J f$. It remains to show that also $P_J P_j f = P_J f$. By Theorem 2.8 we have two different expressions for f:

$$\begin{aligned} f &= P_J f + \sum_{\substack{l \geq J \\ -\infty < k < \infty}} \langle f, h_{l,k} \rangle h_{l,k} \\ &= P_j f + \sum_{\substack{l \geq j \\ -\infty < k < \infty}} \langle f, h_{l,k} \rangle h_{l,k}. \end{aligned}$$

Comparing these to each other we see that

$$\begin{aligned} P_j f &= P_J f + \underbrace{\sum_{\substack{J \leq l < j \\ -\infty < k < \infty}} \langle f, h_{l,k} \rangle h_{l,k}}_{\substack{\text{This part is} \\ \text{orthogonal to } V_J; \\ \text{see Theorem 2.7.}}} . \\ \Longrightarrow P_J P_j f &= P_J \left[P_J f + \sum_{\substack{J \leq l < j \\ -\infty < k < \infty}} \langle f, h_{l,k} \rangle h_{l,k} \right] \\ &= P_J^2 f + 0 \\ &= P_J f \quad \text{(because projection).} \quad \square \end{aligned}$$

III.4 The Haar Detail Operator

Recall that $\{h_{j,k}\}_{k=-\infty}^{\infty}$ is an orthonormal basis for W_j (see Theorem 2.4, page 27). By replacing V_j with W_j we get the *detail operator*:

Definition 4.1. The Haar *detail operator* Q_j of order j is given by

$$Q_j f = \sum_{k=-\infty}^{\infty} \langle f, h_{j,k} \rangle h_{j,k}$$

where $h_{j,k}$ are the Haar wavelets of order j (or scale j).

Theorem 4.2. *The detail Q_j operators have the following properties:*

i) *Q_j is the orthogonal projection of $L^2(\mathbb{R})$ onto the detail space W_j*

ii) $P_{j+1} = P_j + Q_j$

iii) $P_j = P_J + \sum_{J \le l < j} Q_l$, $j > J$

iv) $\lim_{j\to\infty} Q_j f = 0$ *for all* f

v) $\lim_{j\to-\infty} Q_j f = 0$ *for all* f

vi) $Q_j Q_l f = Q_l Q_j f = 0$ *for* $j \neq l$.

Proof.

i) Follows from Theorem 1.6 (page 25) and Theorem 2.4 (page 27)

ii) By Theorem 2.8 (page 29), for all $f \in L^2(\mathbb{R})$

$$\begin{aligned} f &= \sum_{k=-\infty}^{\infty} \langle f, p_{j,k}\rangle p_{j,k} + \sum_{k=-\infty}^{\infty} \langle f, h_{j,k}\rangle h_{j,k} + \sum_{k=-\infty}^{\infty} \sum_{l>j} \langle f, h_{l,k}\rangle h_{l,k} \\ &= P_j f + Q_j f + \text{a part which is orthogonal to } V_{j+1} \end{aligned}$$

$$\Longrightarrow P_{j+1} f = P_j f + Q_j f.$$

iii) Repeat ii) several times.

iv)

$$\begin{aligned} \lim_{j\to\infty} Q_j f &= \lim_{j\to\infty} (P_{j+1} f - P_j f) \quad \text{(by ii))} \\ &= \lim_{j\to\infty} P_{j+1} f - \lim_{j\to\infty} P_j f = f - f = 0. \end{aligned}$$

v)

$$\begin{aligned} \lim_{j\to-\infty} Q_j f &= \lim_{j\to-\infty} (P_{j+1} f - P_j f) \\ &= 0 - 0 = 0. \end{aligned}$$

vi) Follows from Theorem 2.6 on page 28 since $W_j \perp W_l$ for $j \neq l$. □

III.5 Wavelet expansion of Haar Approximation

For numerical computations we first start by approximating f by some function in V_N for some large N.

$$f \approx P_N f. \quad \text{(This is a dyadic step function.)}$$

Then we split $P_N f$ into successively courser and courser "averages" $P_j f$ and the corresponding difference "details":

$$\begin{aligned}
P_N f &= Q_{N-1} f + P_{N-1} f && \text{(first step)} \\
&= Q_{N-1} f + \overbrace{Q_{N-2} f + P_{N-2} f} && \text{(second step)} \\
&= \ldots && \text{(etc.)} \\
&= \underbrace{Q_{N-1} f + Q_{N-2} f + \ldots + Q_J f}_{\text{differences of scale } 2^{-j}} + \underbrace{P_j f}_{\substack{\text{average of} \\ \text{scale } 2^{-J}}} .
\end{aligned}$$

Problem: How do we compute these differences and averages as effectively as possible? (See next chapter for the answer).