Chapter II

Preliminaries

II.1 Hilbert Spaces

Definition 1.1 (Function Spaces)**.** We denote the set of all functions $X \mapsto \mathbb{C}$ by $\mathbb{C}^X$. Here X is an arbitrary set. We define

$$\begin{aligned} f+g: &\quad x \mapsto f(x)+g(x), \quad &(= \text{“sum” of } f \text{ and } g),\\ cf: &\quad x \mapsto cf(x), &(= \text{“product” of } f \text{ and a constant } c \in \mathbb{C}). \end{aligned}$$

This makes $\mathbb{C}^X$ a *vector space.*

Example 1.2. $\ell^2(\mathbb{Z})$ consists of all functions (= sequences in this case) $f : \mathbb{Z} \mapsto \mathbb{C}$ which satisfy the condition

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |f(n)|^2 < \infty.$$

Alternative notation (sequence): $\{f(n)\}_{n=-\infty}^{\infty}$.

In this space we define the *inner product*

$$\langle f, g\rangle = \sum_{n=-\infty}^{\infty} f(n)\overline{g(n)},$$

and the *norm*

$$\|f\| = \sqrt{\langle f, f\rangle} = \Big(\sum_{n=-\infty}^{\infty} f(n)\overline{g(n)}\Big)^{1/2}.$$

These satisfy:

(P) Positivity: $\|f\| > 0$ if $f \neq 0$, $\|0\| = 0$,

(H) Hermitian: $\langle f, g\rangle = \overline{\langle g, f\rangle}$,

(L) Linearity: $\langle \alpha f + \beta g, h\rangle = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$.

Definition 1.3. An *inner product* is a function which maps the pair (f, g) into a number $\langle f, g\rangle$ which has properties (P), (H) and (L) above. A vector space which has an inner product is called a *unitary space* (or *inner product space*). It is a *Hilbert space* if it is, in addition *complete.*

Complete means: $\lim_{n,m\to\infty}\|f_n - f_m\| = 0 \implies$ the limit $\lim_{n\to\infty} f_n = f$ exists.

Lemma 1.4. $\ell^2(\mathbb{Z})$ *is a Hilbert space*

Proof. Analysis II □

Example 1.5. $\ell^2(\mathbb{N})$ is the same as $\ell^2(\mathbb{Z})$, but the "index set" is $\mathbb{N} = \{1, 2, 3, \ldots\}$. The inner product is now

$$\langle f, g\rangle = \sum_{n=1}^{\infty} f(n)\overline{g(n)}.$$

Example 1.6. $L^2(\mathbb{R})$ consists of all measurable functions $\mathbb{R} \mapsto \mathbb{C}$ which satisfy

$$\|f\|_{L^2}^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx < \infty.$$

The inner product is

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx.$$

Two functions are considered to be "equal" if they are equal "almost everywhere" (= ignore the values in a set of measure zero).

Definition 1.7. $f \perp g \iff \langle f, g\rangle = 0$.

Example 1.8. $f = xe^{-x^2}$ is orthogonal to $e^{-x^2} = g$, since

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \int_{-\infty}^{\infty} \underbrace{xe^{-x^2}}_{\text{odd}}\underbrace{e^{-x^2}}_{\text{even}}dx = 0.$$

II.2 Orthonormal Bases

Definition 2.1. A sequence $\{\psi_n\}_{n=1}^{\infty}$ of vectors in a Hilbert space $\mathcal{H}$ is *orthonormal* if

$$\langle \psi_n, \psi_m \rangle = \delta_n^m = \begin{cases} 1, & n = m, \\ 0, & n \neq m. \end{cases}$$

It is an orthonormal *basis* if, in addition

$$\langle \psi_n, f \rangle = 0 \implies f = 0.$$

Thus, *orthonormal* means $\|\psi_n\|^2 = 1$, $\forall n$, and $\psi_n \perp \psi_m$ for $n \neq m$. *Basis means*: only $f = 0$ is orthogonal to every ψ_n.

Definition 2.2. A Hilbert space is *separable* if it has an orthonormal basis.

Theorem 2.3. *The following conditions are equivalent:*

(i) $\{\psi_n\}_{n=1}^{\infty}$ *is an orthonormal basis in* $\mathcal{H}$*.*

(ii) No f *is orthogonal to every* ψ_n*, and*

$$\sum_{n=1}^{k} |c_k|^2 = \|\sum_{n=1}^{k} c_n \psi_n\|_{\mathcal{H}}^2$$

for all k *and all* $c_1, c_2, \ldots, c_k \in \mathbb{C}$*. (This is "Pythagoras theorem".)*

(iii) $\|\psi_n\| = 1$ *for all* n*, and*

$$\|f\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} |\langle \psi_n, f \rangle|^2, \quad f \in \mathcal{H}$$

("Bessel's equality".)

(iv) $\{\psi_n\}_{n=1}^{\infty}$ *is* orthonormal*, and*

$$f = \sum_{n=1}^{\infty} \psi_n \langle \psi_n, f \rangle, \quad f \in \mathcal{H}.$$

Proof. "Analysis II" or "Hilbert spaces". □

Example 2.4.

A) $\psi_n = \frac{1}{\sqrt{T}} e^{2\pi i n t/T}$, $n = 0, \pm 1, \pm 2, \dots$ is an orthonormal basis in $L^2(0,T)$. See Example 1.1 on page 3.

B) $\psi_n = \frac{1}{\sqrt{N}} e^{2\pi i n k/N}$, $n = 0, 1, 2, \dots, N-1$ is an orthonormal basis in $\mathbb{C}^N$. See FFT (course on Fourier Analysis).

Note 2.5. The formulas which involve *sums* in the intro can be interpreted as "expansions with respect to an orthonormal basis". Those involving *integrals* have a different name: They are "resolutions of the identity" which are based on "unitary mappings between two Hilbert spaces".

II.3 Orthogonal Projections

Definition 3.1. Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. A function $A : \mathcal{H} \mapsto \mathcal{K}$ is a *linear operator* if

a) $A(\alpha u) = \alpha A(u), \quad u \in \mathcal{H},\ \alpha \in \mathbb{C}$

b) $A(u + v) = A(u) + A(v), \quad u, v \in \mathcal{H}.$

This operator is *bounded* if there is some constant $M \in \mathbb{R}_+$ so that

$$\|Au\|_{\mathcal{K}} \leq M \|u\|_{\mathcal{H}}, \quad u \in \mathcal{H}.$$

Definition 3.2. A (bounded) linear *functional* on $\mathcal{H}$ is the same as a (bounded) linear operator from $\mathcal{H}$ to $\mathbb{C}$ (i.e., $\mathcal{K} = \mathbb{C}$).

Example 3.3. Let $h \in \mathcal{H}$. Then the mapping $F(u) = \langle u, h \rangle$ $(= h^* u)$ is a bounded linear functional on $\mathcal{H}$.

Proof. Linearity easy. Boundedness follows from Schwartz inequality (see below). □

Theorem 3.4 (Schwartz inequality)**.** *In every Hilbert space $\mathcal{H}$ we have*

$$|\langle f, g \rangle_{\mathcal{H}}| \leq \|f\|_{\mathcal{H}} \|g\|_{\mathcal{H}}$$

where $\|f\|_{\mathcal{H}} = \sqrt{\langle f, f \rangle}$ *and* $\|g\|_{\mathcal{H}} = \sqrt{\langle g, g \rangle}$.

Proof. Analysis II. □

Definition 3.5. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces, and let A be a bounded linear operator $\mathcal{H} \mapsto \mathcal{K}$. Then

$$
\begin{aligned}
i) \quad \mathscr{N}(A) &= \text{the "kernel" of } A \\
&= \text{the "null space" of } A \\
&= \{u \in \mathcal{H} \mid Au = 0\}, \\
ii) \quad \mathscr{R}(A) &= \text{the "range" of } A \\
&= \{y \in \mathcal{K} \mid y = Au \text{ for some } u \in \mathcal{H}\}.
\end{aligned}
$$

Definition 3.6. An operator $P : \mathcal{H} \mapsto \mathcal{H}$ is a *projection* on $\mathcal{H}$ if $P^2 = P$ (that is, if we repeat P two times after each other, then the second time it does nothing). It is *orthogonal* in $\mathscr{N}(P) \perp \mathscr{R}(P)$ i.e., every $u \in \mathscr{N}(P)$ is orthogonal to every $y \in \mathscr{R}(P)$.

Theorem 3.7. *Let $\mathcal{H}$ be a closed subspace of $\mathcal{K}$ (both Hilbert spaces).*

i) There is an orthogonal projection P which maps $\mathcal{K}$ onto $\mathcal{H}$.

ii) Given any $y \in \mathcal{K}$, the vector u in $\mathcal{H}$ which lies closest *to y (in the sense that $\|u - y\|$ is as small as possible) is $u = Py$ (with P as in i)).*

iii) If $\mathcal{H}$ has an orthonormal basis $\{\psi_n\}_{n=1}^{\infty}$, then u in ii) is given by

$$u = \sum_{n=1}^{\infty} \langle y, \psi_n \rangle \psi_n.$$

Proof. Analysis II or Hilbert spaces. ☐

II.4 Finite Fourier Transform

Definition 4.1. $\mathbb{T}$ stands for the real line, where we identify any two points x and y which differ by an integer. Thus, "x is equivalent with y" if $y = x + n$ for some integer $n \in \mathbb{Z}$.

Definition 4.2. The function spaces $C(\mathbb{T})$, $L^p(\mathbb{T})$ $(p = 1, 2, \infty)$ consists of functions which are *periodic* with period one (i.e., $f(x) = f(y)$ if $x - y$ is an integer), and belong locally to C or L^p, etc.

Definition 4.3 (The Finite Fourier Transform)**.** For each $f \in L^1(\mathbb{T})$ we define the (finite) Fourier transform of f by

$$\hat{f}(n) = \int_0^1 e^{-2\pi i n x} f(x) dx, \quad n \in \mathbb{Z}.$$

Theorem 4.4 (Riemann-Lebesgue Lemma)**.** *If* $f \in L^1(\mathbb{T})$ *then* $\hat{f} \in c_0(\mathbb{Z})$ *and* $\|\hat{f}\|_{\ell^\infty} \le \|f\|_{L^1}$, *i.e.,*

i) $|\hat{f}(n)| \le \int_0^1 |f(t)| dt$, $n \in \mathbb{Z}$, *and*

ii) $\hat{f}(n) \to 0$ *as* $n \to \pm\infty$.

Theorem 4.5 (Inversion Theorem A)**.** *If* $f \in L^1(\mathbb{T})$ *and*

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty \quad (\iff \hat{f} \in c_0(\mathbb{Z})),$$

then

$$f(x) = \sum_{n=-\infty}^{\infty} e^{2\pi i n x} \hat{f}(n)$$

for almost all x.

Theorem 4.6 (Inversion Theorem B)**.** *If* $f \in L^1(\mathbb{T})$ *and if for some* $x_0 \in \mathbb{R}$ *we have*

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \left| \frac{f(x) - f(x_0)}{x - x_0} \right| dx < \infty \quad \text{for some } \varepsilon > 0,$$

then

$$f(x_0) = \lim_{N\to\infty} \sum_{n=-N}^{N} e^{2\pi i n x} \hat{f}(n).$$

Theorem 4.7 (Basic Properties)**.** $f \in L^2(\mathbb{T})$, $\tau \in \mathbb{R}$, $k \in \mathbb{Z}$, $k \neq 0$

$$\begin{array}{llcl}
(a) & g(x) = f(x-\tau) & \Longleftrightarrow & \hat{g}(n) = e^{-2\pi i n\tau}\hat{f}(n),\ n \in \mathbb{Z} \\
(b) & g(x) = e^{2\pi i k x} f(x) & \Longleftrightarrow & \hat{g}(n) = \hat{f}(n-k) \\
(c) & g(x) = f(-x) & \Longleftrightarrow & \hat{g}(n) = \hat{f}(-n) \\
(d) & g(x) = \overline{f(x)} & \Longleftrightarrow & \hat{g}(n) = \overline{\hat{f}(-n)} \\
(e) & g(x) = kf(kx) & \Longleftrightarrow & \hat{g}(n) = \begin{cases} \hat{f}(\frac{n}{k}), & \textit{if } \frac{n}{k} \textit{ integer,} \\ 0, & \textit{otherwise} \end{cases} \\
(f) & g \in L^1(\mathbb{T}) \textit{ and } h = f * g & \Longrightarrow & \hat{g}(n) = \hat{f}(n)\hat{h}(n) \\
(g) & \left.\begin{array}{l} f \textit{ abs. cont. and} \\ f' = g \in L^1(\mathbb{T}) \end{array}\right\} & \Longrightarrow & \hat{g}(n) = 2\pi i n \hat{f}(n).
\end{array}$$

Proof. Course on Fourier Analysis. Here

$$(f * g)(x) = \int_0^1 f(x-y)g(y)dy,$$

and "f absolutely continuous and $f' = g$" means that

$$f(x) = f(a) + \int_0^x g(y)dy,$$

where g is locally integrable. □

Theorem 4.8 (Plancherel's Formula)**.** *If $f \in L^2(\mathbb{T})$ then $\hat{f} \in \ell^2(\mathbb{Z})$, and*

$$\|\hat{f}\|^2_{\ell^2(\mathbb{Z})} = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \int_0^1 |f(x)|^2 dx = \|f\|^2_{L^2(\mathbb{T})}.$$

Moreover, every $a_n \in \ell^2(\mathbb{Z})$ is the Fourier transform of some function $f \in L^2(\mathbb{T})$.

Theorem 4.9 (Parseval's Formula)**.** *If $f, g \in L^2(\mathbb{T})$, then*

$$\langle \hat{f}, \hat{g} \rangle_{\ell^2(\mathbb{Z})} = \sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)} = \int_0^1 f(x)\overline{g(x)}dx = \langle f, g \rangle_{L^2(\mathbb{T})}.$$

Theorem 4.10 (L^2-derivatives)**.** *(See page 9 for the definition of $W^{k,2}(\mathbb{T})$). For all $k \in \mathbb{Z}_+$:*

$$f \in W^{k,2}(\mathbb{T}) \iff (2\pi i n)^k \hat{f}(n) \in \ell^2(\mathbb{Z}).$$

II.5 Fourier Integrals

Definition 5.1. If $f \in L^1(\mathbb{R})$, then the Fourier transform $\hat{f}$ of f is given by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i \omega x} f(x) dx, \quad \omega \in \mathbb{R}$$

Theorem 5.2 (Riemann-Lebesgue Lemma). *If $f \in L^1(\mathbb{R})$, then $\hat{f} \in C_0(\mathbb{R})$ and $\|\hat{f}\|_{sup} \le \|f\|_{L^1}$, i.e.,*

i) $|\hat{f}(\omega)| \le \int_{-\infty}^{\infty} |f(x)| dx$, $\omega \in \mathbb{R}$, *and*

ii) $\hat{f}(\omega) \to 0$ *as* $\omega \to \pm\infty$.

Theorem 5.3 (Inversion Theorem A). *If both $f \in L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$ then*

$$f(x) = \int_{-\infty}^{\infty} e^{2\pi i \omega x} \hat{f}(\omega) d\omega$$

for almost all x.

Theorem 5.4 (Inversion Theorem B). *If $f \in L^1(\mathbb{R})$ and if fore some $x_0 \in \mathbb{R}$ we have*

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \left| \frac{f(x) - f(x_0)}{x - x_0} \right| dx \le \infty \quad \textit{for some } \varepsilon > 0,$$

then

$$f(x_0) = \lim_{\Omega \to \infty} \int_{-\Omega}^{\Omega} e^{2\pi i \omega x_0} \hat{f}(\omega) d\omega$$

Theorem 5.5 (Basic Properties). $f \in L^2(\mathbb{T})$, $\tau, \lambda \in \mathbb{R}$, $\lambda \neq 0$

$$
\begin{array}{lll}
(a) \quad g(x) = f(x-\tau) & \Longleftrightarrow & \hat{g}(\omega) = e^{-2\pi i\omega\tau}\hat{f}(\omega) \\
(b) \quad g(x) = e^{2\pi i\tau x} f(x) & \Longleftrightarrow & \hat{g}(\omega) = \hat{f}(\omega-\tau) \\
(c) \quad g(x) = f(-x) & \Longleftrightarrow & \hat{g}(\omega) = \hat{f}(-\omega) \\
(d) \quad g(x) = \overline{f(x)} & \Longleftrightarrow & \hat{g}(\omega) = \overline{\hat{f}(-\omega)} \\
(e) \quad g(x) = \lambda f(\lambda x) & \Longleftrightarrow & \hat{g}(\omega) = \hat{f}(\frac{\omega}{\lambda}) \\
(f) \quad g \in L^1(\mathbb{T}) \text{ and } h = f * g & \Longrightarrow & \hat{h}(\omega) = \hat{f}(\omega)\hat{g}(\omega) \\
(g) \quad g(x) = -2\pi i x f(x) \in L^1(\mathbb{R}) & \Longrightarrow & \hat{f} \in C^1(\mathbb{R}) \text{ and } \hat{g}(\omega) = \hat{f}'(\omega) \\
(h) \quad \left.\begin{array}{l} f \text{ absolutely continuous} \\ \text{and } f' = g \in L^1(\mathbb{R}) \end{array}\right\} & \Longrightarrow & \hat{g}(\omega) = 2\pi i\omega\hat{f}(\omega).
\end{array}
$$

Proof. From the course on Fourier Analysis. f absolutely continuous and $f' = g$ means that

$$f(x) = f(a) + \int_0^x g(y)dy$$

(where g is locally in L^1).

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)g(y)dy. \quad \square$$

Theorem 5.6 (Plancherel's Formula). *If $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, then $\hat{f} \in L^2(\mathbb{R})$, and*

$$\|\hat{f}\|_{L^2}^2 = \int_{-\infty}^{\infty} |\hat{f}(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |f(x)|^2 dx = \|f\|_{L^2}^2.$$

If we drop the condition $f \in L^1(\mathbb{R})$ then we can still define

$$\hat{f}(\omega) = \lim_{M\to\infty} \int_{-M}^{M} e^{-2\pi i\omega x} f(x)dx$$

(where the convergence is in the L^2-sense). After this extension the Fourier transform maps $L^2(\mathbb{R})$ one-to-one onto *$L^2(\mathbb{R})$.*

Theorem 5.7 (Parseval's Formula). *If $f, g \in L^2(\mathbb{R})$, then*

$$\langle \hat{f}, \hat{g} \rangle_{L^2(\mathbb{R})} = \int_{-\infty}^{\infty} \hat{f}(\omega)\overline{\hat{g}(\omega)}d\omega = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx = \langle f, g \rangle_{L^2(\mathbb{R})}.$$

Theorem 5.8 (L^2-derivatives)**.** *(See page 9 for the definition of $W^{k,2}(\mathbb{R})$). For all $k \in \mathbb{Z}^+$:*

i) $f \in W^{k,2}(\mathbb{R}) \iff (2\pi i\omega)^k \hat{f}(\omega) \in L^2(\mathbb{R})$

ii) $(-2\pi i x)^k f(x) \in L^2(\mathbb{R}) \iff \hat{f} \in W^{k,2}(\mathbb{R})$.

II.6 Fourier Series

Definition 6.1. If $\{a_n\}_{n=-\infty}^{\infty} \in \ell^1(\mathbb{Z})$, then the Fourier transform of a_n is given by

$$\hat{a}(\omega) = \sum_{n=-\infty}^{\infty} e^{-2\pi i\omega n} a_n, \quad \omega \in \mathbb{R}.$$

Lemma 6.2. *If $\{a_n\}_{n=-\infty}^{\infty} \in \ell^1(\mathbb{Z})$, then $\hat{a} \in C(\mathbb{T})$, and*

$$\|\hat{a}\|_{\sup} = \sup_{\omega} |\hat{a}(\omega)| \leq \sum_{n=-\infty}^{\infty} |a_n| = \|a\|_{\ell^1}.$$

Theorem 6.3 (Inversion Theorem)**.** *If $\{a_n\}_{n=-\infty}^{\infty} \in \ell^1(\mathbb{Z})$ (or more generally, if $a_n \in \ell^2(\mathbb{Z})$; see below), then*

$$a_n = \int_0^1 e^{2\pi i\omega n} \hat{a}(\omega) d\omega, \quad n \in \mathbb{Z}.$$

Theorem 6.4 (Basic Properties)**.** $a_n \in \ell^1(\mathbb{Z})$, $\tau \in \mathbb{R}$, $k \in \mathbb{Z}$

$$\begin{array}{lllll}
(a) & b_n = a_{n-k} & \iff & \hat{b}(\omega) = e^{-2\pi i\omega k}\hat{a}(\omega) \\
(b) & b_n = e^{2\pi i\tau n} a_n & \iff & \hat{b}(\omega) = \hat{a}(\omega - \tau) \\
(c) & b_n = a_{-n} & \iff & \hat{b}(\omega) = \hat{a}(-\omega) \\
(d) & b_n = \overline{a_n} & \iff & \hat{b}(\omega) = \overline{\hat{a}(-\omega)} \\
(e) & b_n = \begin{cases} a_{n/k}, & n/k \text{ integer} \\ 0, & \text{otherwise} \end{cases} & \iff & \hat{b}(\omega) = k\hat{a}(k\omega),\ (k \in \mathbb{Z}_+) \\
(f) & b_n \in \ell^1(\mathbb{Z}) \text{ and } c = a * b & \implies & \hat{c}(\omega) = \hat{a}(\omega)\hat{b}(\omega) \\
(g) & b_n = -2\pi i n a_n \in \ell^1(\mathbb{Z}) & \implies & \hat{a} \text{ abs. cont. and } \hat{b}(\omega) = \hat{a}'(\omega).
\end{array}$$

Here

$$(a * b)_n = \sum_{k=-\infty}^{\infty} a_{n-k} b_k$$

Theorem 6.5 (Plancherel's Formula)**.** *If* $\{a_n\}_{n=-\infty}^{\infty} \in \ell^1(\mathbb{Z})$ *then*

$$\|\hat{a}\|_{L^2(\mathbb{T})}^2 = \int_0^1 |\hat{a}(\omega)|^2 d\omega = \sum_{n=-\infty}^{\infty} |a_n|^2 = \|a\|_{\ell^2}^2.$$

If $\{a_n\} \notin \ell^1(\mathbb{Z})$*, but it is still true that* $\{a_n\} \in \ell^2(\mathbb{Z})$*, then we can still define*

$$\hat{a}(\omega) = \lim_{N\to\infty} \sum_{n=-N}^{N} e^{2\pi i\omega n} a_n$$

(where the convergence is in L^2*-sense). After this extension the Fourier transform maps* $\ell^2(\mathbb{Z})$ *one-to-one onto* $L^2(\mathbb{T})$*.*

Theorem 6.6 (Parseval's Formula)**.** *If* $\{a_n\}, \{b_n\} \in \ell^2(\mathbb{Z})$*, then*

$$\langle \hat{a}, \hat{b} \rangle_{L^2} = \int_0^1 \hat{a}(\omega)\overline{\hat{b}(\omega)} d\omega = \sum_{n=-\infty}^{\infty} a_n \overline{b_n} = \langle a, b \rangle_{\ell^2}$$

Theorem 6.7 (ℓ^2-moments)**.** *(See page 9 for the definition of* $W^{k,2}(\mathbb{T})$*). For all* $k \in \mathbb{Z}^+$*:*

$$(-2\pi i n)^k a_n \in \ell^2(\mathbb{Z}) \iff \hat{a} \in W^{k,2}(\mathbb{T}).$$

Note 6.8. The Fourier series defined above is almost the same thing as the *inverse* of the Finite Fourier Transform. The only difference is that we have replaced $+2\pi i\omega n$ by $-2\pi i\omega n$.

II.7 Bandlimited Functions

Definition 7.1. A function $f \in L^2(\mathbb{R})$ is *bandlimited* if (f is continuous and) there is a number $\Omega > 0$ such that $\hat{f}(\omega) = 0$ for $|\omega| > \frac{\Omega}{2}$. The smallest such number Ω is called the *bandlimit* of f. Thus, the bandlimit of f is equal to

$$\inf\{\Omega > 0 \mid \hat{f}(\omega) = 0 \text{ for (almost) all } \omega > \Omega\}.$$

Note 7.2. The continuity restriction of f is almost redundant: If $\hat{f}(\omega) = 0$ for $|\omega| > \frac{\Omega}{2}$ then $\hat{f} \in L^1(\mathbb{R})$, and the inverse Fourier transform of $\hat{f}$ is a continuous function (see Theorems 5.2-5.3 on page 17, interchanging the Fourier transform with the inverse Fourier transform) which is almost everywhere equal to f. Thus, if f is *not* continuous, then we can make f continuous by redefining f on a set of measure zero. This function is actually C^∞ (=infinitely many times differentiable; see Theorem 5.8 on page 19).

Theorem 7.3 (The Shannon Sampling Theorem)**.** *If $f \in L^2(\mathbb{R})$ is bandlimited with bandlimit $\Omega > 0$, then f can be written as*

$$f(x) = \sum_{n=-\infty}^{\infty} f(n/\Omega) \frac{\sin(\pi\Omega(x - n/\Omega))}{\pi\Omega(x - n/\Omega)}$$

where the sum converges both uniformly and in $L^2(\mathbb{R})$.

Proof. By property (d) in Theorem 5.5, it suffices to prove this when $\Omega = 1$ (replace $f(x)$ by $g(x) = f(x/\Omega)$). Thus, below we take $\Omega = 1$. By the theory in Section II.4, with the Fourier transform replaced by the inverse Fourier transform, we have for almost all $\omega \in [-1/2, 1/2]$

$$\hat{f}(\omega) = \sum_{n=-\infty}^{\infty} c_n e^{-2\pi i\omega n}, \tag{1}$$

where c_n are the (inverse) Fourier coefficients of $\hat{f}$, i.e.,

$$c_n = \int_{-1/2}^{1/2} e^{2\pi i\omega n} \hat{f}(\omega) d\omega.$$

The convergence of (1) is of L^2-type, and the right-hand side of (1) is a periodic function, whose restriction to $[-1/2, 1/2]$ coincides with the given function $\hat{f}$. However, by Note 7.2 above and by the inversion theorem A on page 17, we have actually $c_n = f(n)$ (since $\hat{f}(\omega) = 0$ for $|\omega| > 1/2$). Thus,

$$\hat{f}(\omega) = \chi_{[-1/2,1/2]} \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i\omega n}, \tag{2}$$

where the convergence is in $L^2[-1/2, 1/2] \implies$ convergence in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. (Here $\chi_{[-1/2,1/2]} = 1$ if $|\omega| \leq 1/2$, and zero otherwise). Multiply this by $e^{2\pi i\omega x}$, and integrate over $[-1/2, 1/2]$. This gives by the Fourier inversion formula (for all $x \in \mathbb{R}$)

$$
\begin{aligned}
f(x) &= \int_{-1/2}^{1/2} e^{2\pi i\omega x} \hat{f}(\omega) d\omega \\
&= \int_{-1/2}^{1/2} e^{2\pi i\omega x} \sum_{n=-\infty}^{\infty} f(n) e^{-2\pi i\omega n} d\omega \\
&\quad \text{(the convergence is absolute, so we may use Fubini's theorem)} \\
&= \sum_{n=-\infty}^{\infty} f(n) \int_{-1/2}^{1/2} e^{2\pi i\omega(x-n)} d\omega \\
&= \sum_{n=-\infty}^{\infty} f(n) \left[\frac{1}{2\pi i(x-n)} e^{2\pi i\omega(x-n)} \right]_{-1/2}^{1/2} \\
&= \sum_{n=-\infty}^{\infty} f(n) \frac{1}{\pi(x-n)} \frac{1}{2i} [e^{\pi i(x-n)} - e^{-\pi i(x-n)}] \\
&= \sum_{n=-\infty}^{\infty} f(n) \frac{\sin(\pi(x-n))}{\pi(x-n)}.
\end{aligned}
$$

The convergence is absolute because of the fact that the sequence on the right-hand side of (2) converges in $L^1(\mathbb{R})$ (this is related to part i) of Theorem 5.2 on page 17, with the direct Fourier transform replaced by the inverse Fourier transform). We also have convergence in L^2 because of the fact that the right-hand side of (2) converges in $L^2(\mathbb{R})$, and the (inverse) Fourier transform preserves convergence in $L^2(\mathbb{R})$; see Theorem 5.6 on page 18. □

Note. This theorem is important in signal processing. It says that if f is bandlimited with bandlimit Ω, then the samples of f at the points $\{n/\Omega\}_{n\in\mathbb{Z}}$ determine f uniquely, and f can be uniquely recreated by its samples.