Chapter I

Introduction

I.1 Analysis and Synthesis of Signals

The classical analysis method in signal processing is *Fourier Analysis.* Typically one performs the following tasks:

A *Analysis Problem*: The original signal is *analyzed* by breaking down the signal into elementary components, that describe the "characteristic properties" of this particular signal.

B *Synthesis Problem*: If we know the elementary components of the signal, how do we reconstruct the signal from these elementary components, or at least construct a reasonable approximation of the original signal which is "close" to the given one in some sense.

Example 1.1 (Finite Fourier Transform)**.** Let $f(t)$ be a piecewise continuously differentiable function on the interval $[0,1]$.

Analysis Step: We compute the *Fourier Coefficients* of f:

$$\hat{f}(n) = \frac{1}{T}\int_0^T e^{-\frac{2\pi i n t}{T}} f(t)dt, \quad n = 0, \pm 1, \pm 2, \ldots$$

These coefficients tell us "how much of a given frequency n" is present in this particular function f.

Synthesis Step: If we know the "elementary components" $\hat{f}(n)$, $n = 0, \pm 1, \pm 2, \ldots$, then we can reconstruct f by means of the formula

$$f(t) = \lim_{N\to\infty} \sum_{n=-N}^{N} e^{\frac{2\pi i n t}{T}} \hat{f}(n).$$

Example 1.2 (Fourier Integral)**.** Let f be a piecewise continuously differentiable function on $\mathbb{R} = (-\infty, \infty)$, satisfying $\int_{-\infty}^{\infty} |f(t)| dt < \infty$.

Analysis Step: We now compute the Fourier transform of f:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-2\pi i \omega t} f(t) dt, \quad \omega \in \mathbb{R}.$$

Again the value $\hat{f}(\omega)$ represents "the amount of oscillation with frequency ω" that is present in f.

Synthesis Step: If we know $\hat{f}(\omega)$ for all $\omega \in \mathbb{R}$, then we can (under suitable assumptions) reconstruct f by means of the formula

$$f(t) = \int_{-\infty}^{\infty} e^{2\pi i \omega t} \hat{f}(\omega) d\omega.$$

For example, this is true if $\int_{-\infty}^{\infty} |\hat{f}(\omega)| d\omega < \infty$.

Example 1.3 (Fourier Series)**.** The original signal is a discrete time signal $\{a_n\}_{n=-\infty}^{\infty}$ satisfying $\sum_{n=-\infty}^{\infty} |a_n| < \infty$.

Analysis Step: We compute the Fourier Transform of a as

$$\hat{a}(\omega) = \sum_{n=-\infty}^{\infty} e^{2\pi i \omega n} a_n.$$

This is a periodic function with period 1. It is still true that $\hat{a}(\omega)$ somehow represents the amount of oscillation of $\{a_n\}$ of frequency ω.

Synthesis Step: If we know $\hat{a}(\omega)$ then we can recover the original signal by means of the formula

$$a_n = \int_0^1 e^{2\pi i \omega n} \hat{a}(\omega) d\omega.$$

I.2 Compression of Signals

It is possible to add another step:

Compression Step: After performing the analysis we throw away "the least significant part" of the elementary component. This reduces the amount of data that needs to be stored. This makes the synthesis more difficult, and in most cases it is no longer possible to synthesize the original signal from the stored data. However, we may still obtain a reasonable approximation of the original signal.

Example 2.1 (Finite Fourier Transform)**.** *Compression Step*: Choose some tolerance $\varepsilon > 0$ and throw away all Fourier coefficients satisfying $\hat{f}(n) < \varepsilon$. This leaves a finite number of coefficients $\hat{f}(n)$. The reconstruction formula now becomes

$$f(t) \approx \sum_{|\hat{f}(n)|>\varepsilon} e^{\frac{2\pi int}{T}} \hat{f}(n),$$

where the sum is a finite sum.

Example 2.2 (Fourier Integrals)**.** *Compression Step*: It can be shown that $\hat{f}(\omega) \to 0$ as $\omega \to \infty$. Choose some number $\varepsilon > 0$ and some integer N, and keep only the values

$$\hat{f}(\varepsilon k), \quad -N \leq k \leq N, \quad (k = \text{integer}),$$

and throw away the remaining values of $\hat{f}$. We are then down to a finite set of data. The reconstruction formula is obtained from the integral formula on page 4 by approximating the integral e.g. by a Riemann sum:

$$f(t) \approx \sum_{k=-N}^{N} e^{2\pi i(\varepsilon k)t} \hat{f}(\underbrace{\varepsilon k}_{\omega_k}) \underbrace{\varepsilon}_{\Delta\omega}.$$

Example 2.3 (Fourier Series)**.** *Compression Step*: We again discretize $\hat{a}$: Fix some $N > 0$ and keep the values $\hat{a}(k/N)$, $k = 0, 1, 2, \ldots, N-1$, and throw away the rest. Then the approximative reconstruction formula becomes (if we use a Riemann sum approximation of the integral):

$$a_n \approx \sum_{k=0}^{N-1} e^{2\pi i(\frac{k}{N})n} \hat{a}(\underbrace{\frac{k}{N}}_{\omega_k}) \underbrace{\frac{1}{N}}_{\Delta\omega}.$$

I.3 Wavelets

The "compression" methods described above have several disadvantages:

- They are *not* very *efficient*. The amount of data that we need to store is quite large compared to the achieved accuracy of approximation.

- If the original signal is defined on $\mathbb{R}$, then we need an extra "cutoff" approximation if we want to use a finite Fourier transform, or an extra "discretization" step if we want to use a Fourier series. This introduces additional errors and additional work.

All the Fourier examples that we have considered so far use an analysis step where the elementary components consist of either a *sequence* or a *function of one variable* (= frequency).

In wavelet analysis we use *two variables*: "time" and "scale". The *time* variable indicates which point in time we are interested in (= the original argument of the given signal, sometimes interpreted as "place" instead of time). The *scale* variable resembles the frequency variable in the sense that it tells us "how fast things change". It separates "slow changes" from "fast changes".

Like Fourier transforms wavelets come in many different settings:

I.3.1 Continuous Time and Scale

f is a given function of a continuous variable $u \in \mathbb{R}$.

Analysis Step: The wavelet transform of f is given by

$$\tilde{f}(s,t) = \int_{-\infty}^{\infty} f(u)\bar{\psi}_{s,t}(u)du, \tag{1}$$

where $\psi_{s,t}$ is the *analyzing wavelet*, which has been constructed from a *single mother wavelet* ψ by *translations* (determined by the time variable t) and *dilations* (determined by the scale variable s), so that $\psi_{s,t}(u) = s^{1/2}\psi(s(u-t))$. Dilation = "uttöjning".

Synthesis Step:

$$f(u) = \int_{0}^{\infty}\int_{-\infty}^{\infty} \psi^{s,t}(u)\tilde{f}(s,t)dtds, \tag{2}$$

where $\psi^{s,t}$ is the *synthesizing* wavelet constructed in a similar manner, as $\psi^{s,t}(u) = s^{1/2}\tilde{\psi}(s(u-t))$

I.3.2 Continuous Time, Discrete Scale

f is still a continuous function of a continuous variable $u \in \mathbb{R}$, but now s is discrete.

Analysis Step: The formula for $\tilde{f}(s,t)$ remains the same, but the scale variable only takes discrete values $s_0, s_1, s_2, \ldots$, instead of a continuous set of values.

Synthesis Step: The integral with respect to t remains the same, but the integral with respect to the scale variable is replaced by a sum.

I.3.3 Discrete Time and Scale

The original function f is still defined on all of $\mathbb{R}$ (it is not a sequence but a continuous time signal), but now, both t and s are discrete. Both the integral with respect to t and with respect to s are replaced by sums.

I.3.4 Discrete Time, Scale and Signal

Here the given signal is a *sequence*, not a function of a continuous time variable. This means that now also the integral over the variable u in formula (1) becomes a sum. (In I.3.1 - I.3.3 we still used the integral in (1) to define $\tilde{f}(s,t)$, even if s and/or t was discrete.).

In this course we focus on cases where *both the time and the scale is discrete*, and the original *signal* is either a function of continuous time, or a sequence of discrete time. If the original signal time u is continuous, then we have to start by computing a number of integrals of the type (1), but the rest of the computations are purely algebraic. (We do, however, use Fourier Analysis in many of the proofs of the main theorems.)

I.4 Generalized Fourier Series

The mathematical foundation of the theory consists of "generalized Fourier series". These are described in more detail in two other courses, namely "Analysis II" and "Hilbert Spaces". Lecture notes for the course "Analysis II" are found on the course homepage:

`http://www.abo.fi/fak/mnf/mate/kurser/analys2.`

The idea is the following: We have a Hilbert space $\mathcal{H}$ (= a vector space with a complete inner product $\langle \cdot, \cdot \rangle$). In this space we have a complete orthonormal basis $\{\varphi_n\}_{n=0}^{\infty}$.

"Orthonormal" means that

$$\langle \varphi_n, \varphi_n \rangle = 1 \quad \forall n \in \mathbb{Z}_+ = 0, 1, 2, \ldots,$$
$$\langle \varphi_n, \varphi_k \rangle = 0 \quad \text{if } n \neq k.$$

Analysis Step: Given a vector $f \in \mathcal{H}$, we compute the "generalized Fourier coefficients"

$$\hat{f}(n) = \langle f, \varphi_n \rangle, \quad n \in \mathbb{Z}_+.$$

Here $\hat{f}(n)$ is the projection of f in the direction φ_n.

Synthesis Step: We reconstruct f from its Fourier coefficients by taking

$$f = \sum_{n \in \mathbb{Z}_+} \hat{f}(n) \varphi_n.$$

Compression Step: Keep only the "large" coefficients $\hat{f}(n)$, and throw away the small ones.

The *Magic of Wavelets*: It's all about **how to choose the basis vectors φ_n in a clever way**, so that the compression is as efficient as possible!

I.5 Notations

$\mathbb{R}$	$(-\infty, \infty)$
$\mathbb{C}$	Complex plane
$\mathbb{Z}$	$0, \pm 1, \pm 2, \ldots$
$\mathbb{Z}_+$	$0, 1, 2, \ldots$
$\mathbb{T}$	See section II.4 on page 14
$\ell^1(\mathbb{Z})$	Sequences $\{a_n\}_{n\in\mathbb{Z}}$ satisfying $\|a\|_{\ell^1} = \sum_n \|a_n\| < \infty$
$\ell^2(\mathbb{Z})$	Sequences $\{a_n\}_{n\in\mathbb{Z}}$ satisfying $\|a\|_{\ell^2} = (\sum_n \|a_n\|^2)^{1/2} < \infty$
$\ell^\infty(\mathbb{Z})$	Bounded sequences $\{a_n\}_{n\in\mathbb{Z}}$ satisfying $\|a\|_{\ell^\infty} = \sup_n \|a_n\| < \infty$
$c_0(\mathbb{Z})$	Sequences $a_n \to 0$ as $n \to \pm\infty$
$\mathbb{C}^X$	all $\mathbb{C}$-valued functions on X
$L^1(\mathbb{R})$	all "measurable" functions φ satisfying $\|\varphi\|_{L^1} = \int_{\mathbb{R}} \|\varphi(x)\| dx < \infty$
$L^2(\mathbb{R})$	all "measurable" functions φ satisfying $\|\varphi\|_{L^2} = (\int_{\mathbb{R}} \|\varphi(x)\|^2 dx)^{1/2} < \infty$
$L^\infty(\mathbb{R})$	all "measurable" and "essentially bounded" functions φ, with $\|\varphi\|_{L^\infty} = \operatorname{essup}_{x\in\mathbb{R}} \|\varphi(x)\| < \infty$
$C(\mathbb{R})$	Continuous functions on $\mathbb{R}$
$C_0(\mathbb{R})$	Continuous functions on $\mathbb{R}$ which tend to zero at $\pm\infty$
$L^p(\mathbb{T})$	($p = 1, 2$ or ∞). Periodic functions on $\mathbb{R}$ with period 1, which belongs to $L^p([0,1])$.
$C(\mathbb{T})$	Continuous periodic functions on $\mathbb{R}$ with period 1
$C^n(\mathbb{R})$	n times continuously differentiable functions on $\mathbb{R}$
$C^n(\mathbb{T})$	Functions in $C^n(\mathbb{R})$ which are periodic with period 1
$W^{n,2}(\mathbb{R})$	Functions which are $n-1$ times continuously differentiable, and $f^{(k)} \in L^2(\mathbb{R})$ for $k = 0, 1, 2, \ldots, n$, where $f^{(n)}$ is the distribution derivative of $f^{(n-1)}$, i.e., $f^{(n-1)}(x) = f^{n-1}(0) + \int_0^x f^{(n)}(y) dy$
$W^{n,2}(\mathbb{T})$	Functions which are locally in $W^{n,2}(\mathbb{R})$ and which are periodic with period one.