Note: If $E = F = \mathbb{R}^n$ or $\mathbb{C}^n$, then we know from matrix algebra: T.f.c.a. equivalent:

(i) A is invertible
(ii) A is one-to-one (full row rank)
(iii) A is onto (full column rank)
(iv) A has a left-inverse B, i.e., $BA = I$
(vi) A has a right-inverse C, i.e. $AC = I$.
(vii) $\det(A) \neq 0$.

In infinite dimensions none of these are equivalent. Determinants cannot even be defined.

7.9 Ex. Take S, S^* from Ex. 7.2d. Then

$$S^* S = I, \qquad S S^* \neq I, \text{ so}$$

neither is invertible. Out of these S is one-to-one and S^* is onto.
= injective

In control theory the following result is known under the name "small gain thm":

7.10 Thm. Let E be a Banach space, and let $A = \mathcal{L}(E)$ $(= \mathcal{L}(E;E))$. If $\|A\| < 1$, then $I - A$ is invertible, and

$$\textcircled{*} \quad (I-A)^{-1} = \sum_{n=0}^{\infty} A^n \quad \text{(convergence in the Banach space } \mathcal{L}(E)\text{)}.$$

Here $A^n = \underbrace{A\,A\,A\,A \cdots A}_{n \text{ factors}}$ $(n \geq 1)$

$A^0 = I$ (by definition)

Equivalent way of writing $\textcircled{*}$:

$$(I-A)^{-1} = \lim_{n\to\infty} (I + A + A^2 + A^3 + \cdots + A^n).$$

Note: True under weaker cond: Spectral radius < 1)

Proof. (simpler than in the book).

First claim: $\lim_{n\to\infty} \sum_{k=0}^{n} A^k = \lim_{n\to\infty} T_n$ exists.

(thus, $T_n = I + A + A^2 + \cdots + A^n$.)

Proof. Since $L(E)$ is a Banach space, it suffices to show that T_n is a Cauchy seq. in $L(E)$. We have (for $n \geq m$)

$$\|T_n - T_m\| = \|A^m + A^{m+1} + \cdots + A^n\|$$
$$= \|A^m (I + A + \cdots + A^{n-m})\| \quad \text{(by Cor. 7.7)}$$
$$\leq \|A\|^m (1 + \|A\| + \|A\|^2 + \cdots \|A\|^{n-m})$$
$$\leq \|A\|^m \textstyle\sum_{k=1}^{\infty} \|A\|^k \quad \text{(geometric series)}$$
$$= \|A\|^m \frac{1}{1-\|A\|}. \quad \text{This} \to 0 \text{ as } m \to \infty$$

$\Rightarrow$ Cauchy seq $\Rightarrow$ convergence.

Thus, $T = \sum_{n=1}^{\infty} A^k \in L(E)$.

Claim 2: $T(I-A) = (I-A)T = I$.

Proof:
$$(I-A)T = (I-A) \lim_{n\to\infty} \sum_{k=0}^{n} A^k \quad \text{(move } (I-A) \text{ inside limit)}$$
$$= \lim_{n\to\infty} \left(\sum_{k=0}^{n} A^k - \sum_{k=1}^{n+1} A^k \right)$$
$$= \lim_{n\to\infty} (I - A^{n+1}) = I$$
$$\|A^{n+1}\| \leq \|A\|^{n+1} \to 0 \text{ as } n \to \infty.$$

The same computation with $(I-A)$ on the other side gives $T(I-A) = I$. So $T = (I-A)^{-1}$ ☐

7.11a Cor. Let E be a Banach space, and $A \in L(E)$. If $\|I - A\| < 1$, then A is invertible, and

$$A^{-1} = \sum_{n=0}^{\infty} (I-A)^n$$

Proof. Apply Thm 7.10 with A replaced by $B = (I-A)$. ☐

7.11b Cor. The set of all invertible operators in $L(E)$ is open in $L(E)$.

Meaning that for every $A \in L(E)$ which is invertible there is an $\varepsilon > 0$ such that all $B \in L(E)$ with $\|B - A\| < \varepsilon$ are invertible. We can actually take $\boxed{\varepsilon = 1/\|A^{-1}\|}$.

Proof. Let A be invertible, and let $\|B - A\| < \frac{1}{\|A^{-1}\|}$. Then

$$B = A(A^{-1}B) = AC, \text{ where}$$

$$\|I - C\| = \|I - A^{-1}B\| = \|A^{-1}(A - B)\| \leq \|A^{-1}\| \|A - B\| < 1.$$

Thus, by Cor. 7.11a, C is invertible. We claim that $C^{-1}A^{-1}$ is an inverse to B. This is true because

$$C^{-1}A^{-1}B = C^{-1}C = I \quad \text{and}$$
$$BC^{-1}A^{-1} = AA^{-1}BC^{-1}A^{-1} = ACC^{-1}A^{-1} = AA^{-1} = I.$$

"adjungerad" eller "dual",

VII.6 Adjoint (Dual) Operators

In matrix theory the Hermitean adjoint A^* is defined to be $A^* = \bar{A}^T$ (the transpose of the complex conjugate of A). If A is of dimension $m \times n$, then for all $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$,

$$(Ax, y) = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} A_{ij} x_j \right) \overline{y_i}$$

$$= \sum_{j=1}^{m} x_j \overline{\left(\sum_{i=1}^{n} \bar{A}_{ij} y_i \right)} = (x, A^* y).$$

In infinite dimensions we use this as a definition of A^*:

7.13 Thm. Let $A \in \mathcal{L}(E; F)$ with E and F Hilbert spaces. Then there is a unique operator $A^* \in \mathcal{L}(F; E)$ so that

$$\circledast \quad (Ax, y)_F = (x, A^* y)_E \quad \forall x \in E \ \forall y \in F.$$

Moreover, $\|A^*\| = \|A\|$, and $(A^*)^* = A$.

Note: $A: E \to F$ and $A^*: F \to E$!

Proof: A) Uniqueness: Assume that
$$(Ax, y) = (x, By) = (x, Cy) \quad \forall x \in E, \forall y \in F.$$
Then $By - Cy$ is orthogonal to E, i.e.,
$$By - Cy = 0 \quad \forall y \in F \Rightarrow B = C.$$

B) definition of A^*: Fix $y \in F$. The mapping $x \mapsto F(x) = (Ax, y)$ is
i) linear since A is linear and $z \mapsto (z, y)$ is linear,
ii) bounded since
$$|(Ax, y)_F| \leq \|Ax\|_F \|y\|_F \leq \|A\| \|x\|_E \|y\|_F.$$

Thus, $F \in E^*$. By Riesz' representation thm, there is a unique $z \in E$ so that $F(x) = (x, z)$. Clearly z depends on $y \in F$. Let us denote $z = A^* y$.

Claim 1: The mapping $y \mapsto A^* y$ is linear.
Proof easy (see book).

Claim 2: The mapping $y \mapsto A^* y$ is bounded.
Proof. Apply $\circledast$ with $x = A^* y$ to get

$$\|A^* y\|^2 = (A^* y, A^* y) = (y, AA^* y)$$

$$\leq \|y\| \|AA^* y\| \leq \|y\| \|A\| \|A^* y\|.$$

If $A^*y \neq 0$, then we can divide by $\|A^*y\|$ and get

$$\|A^*y\| \le \|A\| \|y\|.$$

Trivially, this is also true if $A^*y = 0$. Thus, A^* is bounded, and $\|A^*\| \le \|A\| \|y\|$.

C) $\underline{(A^*)^* = A}$: True because $\forall x \in E$ and $\forall y \in F$,

$$(A^*y, x) = (y, (A^*)^*x) \quad \text{and}$$

$$(y, Ax) = \overline{(Ax, y)} = \overline{(x, A^*y)} = (A^*y, x).$$

Thus $(y, (A^*)^*x) = (y, Ax)$ $\forall y \in F$ $\forall x \in E$, and by part A), $A = (A^*)^*$.

D) $\|A^*\| = \|A\|$. True because by B), $\|A^*\| \le \|A\|$ and $\|A\| = \|(A^*)^*\| \le \|A^*\|$.

<u>7.13a Defn.</u> A^* is the <u>adjoint</u> (or <u>dual</u>) operator of A.

<u>7.14a Ex.</u> The adjoint of the multiplication operator $(M_f x)(t) = f(t)x(t)$, $0 < t < 1$ in Ex. 7.2a is the multiplication operator $(M_{\bar f} x)(t) = \overline{f(t)}\, x(t)$, $0 < t < 1$.

Proof easy. In particular, $(M_f)^* = M_f \Leftrightarrow$ f is real-valued.

<u>7.14b Ex.</u> The adjoint of the integral operator

$$(Tx)(t) = \int_a^b k(t,s)\, x(s)\, ds, \quad c < t < d$$

is the integral operator (over)

$$(T^*y)(s) = \int_c^d y(t)\, \overline{k(t,s)}\, dt, \quad a < s < b.$$

(note that we this time integrate with respect to the <u>first</u> variable and not the second).

In particular, if $a = c$ and $b = d$ so that $E = F = L^2(a,b)$, then $T = T^*$ if and only if $k(s,t) = \overline{k(t,s)}$ for all $s, t \in (a,b)$.

Proof easy.

<u>7.14c Ex:</u> The adjoint of the differentiation operator $Tf = f'$ in Ex. 7.2c is not defined (at this stage) since this operator is not bounded, and its domain is not all of $L^2(-1,1)$.

<u>7.14d Ex.</u> The adjoint of the right shift S in Ex. 7.2d is the left shift S^*.

<u>Proof</u>: Let T be the left shift defined in 7.2d (denoted by S^* there). We claim that this is the adjoint of S. To prove this it is enough to show that for all $x \in \ell^2$ and $y \in \ell^2$,

$$(Sx, y) = (x, Ty),$$

because of the uniqueness part of Thm. 7.13.

This is easy: $\leftarrow$ right shift of x; $(Sx)_0 = 0$

$$(Sx, y) = \sum_{i=0}^{\infty} (Sx)_i \overline{y_i} = \sum_{i=1}^{\infty} x_{i-1} \overline{y_i}$$

$$= \sum_{j=0}^{\infty} x_j \overline{y_{j+1}} = \sum_{j=0}^{\infty} x_j \overline{(Ty)_j}$$

$$= (x, Ty). \quad \square$$

7.15 Thm Adjoint operators have the following properties:

(i) $(A^*)^* = A$
(ii) $\|A^*\| = \|A\|$
(iii) $(AB)^* = B^*A^*$
(iv) $(\lambda A + \mu B)^* = \bar{\lambda} A^* + \bar{\mu} B^*$
(v) A is invertible if and only if A^* is invertible, and in this case $(A^{-1})^* = (A^*)^{-1}$.

Proof. (i)–(ii): See Thm 7.13
(iii) For all $x \in H$ and $y \in H$,

$$(x, (AB)^* y) = (ABx, y) = (Bx, A^* y)$$
$$= (x, B^* A^* y).$$

Since this is true for all x and y we have (why?)

$$(AB)^* = B^* A^*.$$

(iv) Another simple computation analogous to the one above

(v) Let A be invertible. Then, for all x and y,

$$(x, y) = (AA^{-1}x, y) = (x, (AA^{-1})^* y)$$
$$= (x, (A^{-1})^* A^* y).$$

Thus $(A^{-1})^* A^* = I$. In the same way we prove that $A^*(A^{-1})^* = I$. Thus, A^* is invertible with inverse $(A^{-1})^*$. To get the converse direction we apply the same argument with A replaced by A^* and A^* replaced by $(A^*)^* = A$.

VII. 7 Selfadjoint (Hermitean) Operators

7.17 Defn. An operator $A \in \mathcal{L}(H)$ is self-adjoint ("självadjungerad") or Hermitean ("Hermitisk") iff $A^* = A$, or symmetric in the real case.

Note: Maps H into itself. In the matrix case this means that A is a square matrix.

7.18 Thm: For every self-adjoint operator,

$$\|A\| = \sup_{\|x\| \le 1} |(Ax, x)|.$$

Note: This simplifies the comput. of $\|A\|$ since (Ax, x) is much easier to compute than $\|Ax\|$, for $x \in H$.

Numerically, this means that we maximize and minimize the function (Ax, x), and take $\|A\|$ = the larger of $\max_{\|x\|=1} (Ax, x)$

and $-\min_{\|x\| \le 1} (Ax, x)$. Note that (Ax, x) is real.

since $(Ax,x) = (x, A^*x) = \overline{(x, Ax)} = \overline{(Ax,x)}$.

Proof: (Easy part): If $\|x\| \le 1$, then

$$|(Ax,x)| \le \|Ax\|\,\|x\| \le \|A\|\,\|x\|^2 \le \|A\|, \text{ so}$$

$$\|A\| \ge \sup_{\|x\|\le 1} |(Ax,x)|.$$

Converse part (more difficult): Denote

$$m = \sup_{\|x\|\le 1} |(Ax,x)|.$$

Then, for all $x \in H$, (put $y = \frac{x}{\|x\|}$)

$$|(Ax,x)| = (\|x\|\,Ay, \|x\|\cdot y) = \|x\|^2 (Ay,y) \le m\,\|x\|^2. \quad (*)$$

We "polarize" this function:

$$(A(x+y), x+y) = (Ax,x) + (Ax,y) + (Ay,x) + (Ay,y),$$

where $(Ay,x) = (y, A^*x) = (y, Ax) = \overline{(Ax,y)}$,

so

$$(A(x+y), x+y) = (Ax,x) + (Ay,y) + 2\,\mathrm{Re}\,(Ax,y).$$

In the same way:

$$(A(x-y), x-y) = (Ax,x) + (Ay,y) - 2\,\mathrm{Re}\,(Ax,y).$$

Subtract $\Rightarrow$ $4\,\mathrm{Re}\,(Ax,y) = (A(x+y), x+y) - (A(x-y), x-y)$

(by $(*)$) $\le m\left(\|x+y\|^2 + \|x-y\|^2\right)$ (Parall. identity)

$$= 2m\left(\|x\|^2 + \|y\|^2\right).$$

Thus, $\boxed{\mathrm{Re}\,(Ax,y) \le \frac{m}{2}\left(\|x\|^2 + \|y\|^2\right)}$

(if A is self-adjoint).

Take an arbitrary $x \in H$ with $Ax \ne 0$. Then $x \ne 0$. Put $y = \frac{\|x\|}{\|Ax\|} Ax$. Then

$$(Ax, y) = \left(Ax, \frac{\|x\|}{\|Ax\|} Ax\right) = \frac{\|x\|}{\|Ax\|}\|Ax\|^2$$

$= \|x\|\,\|Ax\|$, so $(Ax,y) > 0$, and

$$\|Ax\| = \frac{1}{\|x\|}(Ax,y) \le \frac{1}{\|x\|}\frac{m}{2}\Big(\|x\|^2 + \underbrace{\|y\|^2}_{=\|x\|^2}\Big)$$

$$= m\,\|x\|.$$

Thus, $\|A\| = \sup_{\|x\|\le 1}\|Ax\| \le m$.

7.19 Ex. This is not true for non-self-adjoint operators. For example, let T = a 90° rotation in the plane. This is a unitary operator, i.e., $\|Tx\| = \|x\|\ \forall x \in \mathbb{R}^2$, so $\|T\| = 1$, but $Tx \perp x$, so $(Tx,x) = 0\ \forall x$. Thus

$$1 = \|T\| \ne \sup_{\|x\|\le 1} |(Tx,x)| = 0.$$

Lemma: Let H be a Hilbert space. A projection operator in H is orthogonal if and only if it is self-adjoint.

Proof: By Homework 23, $N(P) = R(P^*)^\perp$, so $N(P) \perp R(P)$ if $P = P^*$, i.e., $P = P^* \Rightarrow P$ is orthogonal. Conversely, if P is orthogonal, then $R(P) \perp N(P) = R(Q)$, so for all $x, y \in H$

$$(Px, y) = (Px, Py + Qy) = (Px, Py) =$$

$$(Px + Qx, Py) = (x, Py) \implies P^* = P. \quad \square$$

VII. 8 The spectrum of an operator

In matrix theory we say that $\lambda \in \mathbb{C}$ is an eigenvalue of $A \in \mathbb{C}^{n\times n}$ and $v \in \mathbb{C}^n$ is an eigenvector if $x \neq 0$ and

$$Ax = \lambda x \qquad (\text{i.e. } Ax \overset{\text{parallell}}{\upuparrows} x \text{ and } x \neq 0).$$

Then $(A-\lambda I)x = 0$, so $(A-\lambda I)$ is not one-to-one, hence not invertible.

Note: only square matrices have eigenvalues and eigenvectors, since Ax and x must belong to the same space.

In the infinite-dimensional case it is possible that $(A-\lambda I)$ is one-to-one but not onto, hence we need a more complicated definition. (see examples on p 58).

7.20a Defn. The spectrum $\sigma(A)$ of an operator $A \in \mathcal{L}(H)$ is the set of points $\lambda \in \mathbb{C}$ for which $\lambda I - A$ is not invertible.

Different possibilities:
- A) A is not one-to-one (possible)
- B) A is not onto (possible)
- C) A is one-to-one and onto, but it does not have a bounded inverse (Cf. Defn. 7.8) We shall see later that this is impossible (when H is a Hilbert space).

(if A is bounded, or more generally, A is closed)

7.20b Defn. λ is an eigenvalue ("egenvärde") of $A \in \mathcal{L}(H)$ if $\lambda I - A$ is not one-to-one, i.e., there is some nonzero x for which $(\lambda I - A)x = 0$, or equivalenty, $Ax = \lambda x$. We call this vector x an eigenvector of A ("egenvektor").

7.21 Ex: An operator with no eigenvalues.
We look at the multiplication operator $(M_\phi x)(t) = \phi(t)x(t)$, $0<t<1$ in $L^2(0,1)$ (see Ex. 7.2a). Here $\phi \in C[0,1]$. We denote the range of ϕ (restricted to $[0,1]$) by $\phi([0,1])$.

For all $\lambda \in \mathbb{C}$, $((\lambda I - M_\phi)x)(t) = \lambda x(t) - \phi(t)x(t)$

$$= [\lambda - \phi(t)]x(t), \quad 0<t<1.$$

This is still a multiplication operator $M_{\lambda-\phi}$. Obviously, if $\lambda \notin \phi([0,1])$, then $\lambda - \phi(t) \neq 0$ for all $t \in [0,1]$, and $\frac{1}{\lambda - \phi(t)} \in C[0,1]$.

If we define $g(t) = \frac{1}{\lambda - \phi(t)}$, then

$$(M_g M_{\lambda-\phi} x)(t) = g(t)[(\lambda - \phi(t))x(t)] = x(t), \quad 0<t<1,$$

and also $M_{\lambda-\phi} M_g x = x$. Thus,

$\lambda I - M_\phi$ is invertible, and $(\lambda I - M_\phi)^{-1} = M_g$.
This shows that

$$\boxed{\text{no point outside the range of } \phi \text{ belongs to the spectrum of } M_\phi.}$$

What about points in the range?

Answer: They all belong to the spectrum. They may or may not be eigenvalues. They are eigenvalues if the function takes this value in a whole interval (i.e., $\phi(t) = \lambda$ for all t in some (small) interval), otherwise they are not. But they still belong to the spectrum. For details, see the book.

7.21a Ex. The right-shift operator S on p. 58 has no eigenvalues, but the spectrum of S is the full closed unit disk $\bar{D} = \{z \in \mathbb{C} \mid |z| \le 1\}$.

Proof: Suppose x is an eigenvector ($\neq 0$), and $Sx = \lambda x$. Then

$$Sx = \{0, x_1, x_2, x_3, \dots\}$$
$$\lambda x = \{\lambda x_1, \lambda x_2, \lambda x_3, \dots\}, \text{ so}$$
$$\lambda x - Sx = \{\lambda x_1, \lambda x_2 - x_1, \lambda x_3 - x_2, \dots\}$$

If this is $= 0$, then

$\lambda x_1 = 0$
$\lambda x_2 = x_1$
$\lambda x_3 = x_2$
etc.

If $\lambda = 0 \Rightarrow$ all $x_i = 0$
If $\lambda \neq 0$, then
$x_1 = 0/\lambda = 0$
$x_2 = x_1/\lambda = 0$, etc.

Thus, in both cases, $x = 0$, so S has no eigenvectors.

That the spectrum of S is the full unit disk is a homework.

7.21b For the left-shift operator S^* on p. 58 every $\lambda \in D = \{z \in \mathbb{C} \mid |z| < 1\}$ is an eigenvector, and the spectrum of S^* is the closed unit disk.

Proof: Take $\lambda \in D$, and try to find a nonzero eigenvector

$$x = (x_1, x_2, \dots)$$
$$S^* x = (x_2, x_3, x_4, \dots)$$
$$\lambda x - S^* x = (\lambda x_1 - x_2, \lambda x_2 - x_3, \lambda x_3 - x_4, \dots) = 0$$

gives

$$x_2 = \lambda x_1$$
$$x_3 = \lambda x_2$$
$$x_4 = \lambda x_3 \quad \text{etc.}$$

For example, we can take $x_1 = 1$, $x_2 = \lambda$, $x_3 = \lambda^2$, $x_4 = \lambda^3, \dots$

This vector belongs to l^2, since

$$\sum_{k=1}^{\infty} |\lambda^k|^2 = \sum_{k=1}^{\infty} (|\lambda|^2)^k = \frac{1}{1-|\lambda|^2} < \infty.$$

Thus, we have found a nonzero eigenvector for every $\lambda \in \mathbb{C}$.

The rest of this example is a homework.

7.22 Thm The spectrum $\sigma(A)$ of a bounded linear operator A is a closed and bounded subset of $\mathbb{C}$. More precisely, every $\lambda \in \sigma(A)$ satisfies $|\lambda| \le \|A\|$.

Later: we shall also see that it is always nonempty.

Proof. We must show that
i) the spectrum is closed
ii) the spectrum is bounded.
We begin with ii): If $|\lambda| > \|A\|$, then

$$I - \frac{1}{\lambda} A \quad \text{is invertible by Thm 7.10,}$$

hence $\lambda (I - \frac{1}{\lambda} A) = (\lambda I - A)$ is invertible,
hence $\lambda \notin \sigma(A)$.

Next we take i): $\sigma(A)$ closed $\Leftrightarrow$ the complement is open. Let $\lambda \notin \sigma(A)$, so that $(\lambda I - A)$ is invertible. By Corollary 7.11b, if $B \in \mathcal{L}(H)$ and $\|B - (\lambda I - A)\| < \|(\lambda I - A)^{-1}\|$, then B is invertible. Choose B to be $B = \mu I - A$, with $|\mu - \lambda| < \|(\lambda I - A)^{-1}\|$. Then $\|B - A\| = \|(\mu - \lambda) I\| = |\mu - \lambda| < \|(\lambda I - A)^{-1}\|$, so $B = (\mu I - A)$ is invertible and $\mu \notin \sigma(A)$. $\square$

VII.9 Infinite Matrices.

Formally, in $\mathbb{C}^n$ or $\mathbb{C}^n$ we get the matrix representation of an operator $\mathbb{C}^n \to \mathbb{C}^m$ like this: Take the basis $\{e^k\}$ in $\mathbb{C}^n$ and ϕ^k in $\mathbb{C}^m$ where

$$e^k = \underbrace{\{0, 0, 0, 1, 0, \dots\}}_{n \text{ numbers}}$$
↑ k-th position,

$$\text{and } \phi^k = \underbrace{\{0, 0, 0, 1, 0, 0\}}_{m \text{ numbers}}$$

Then every $x \in \mathbb{C}^n$ can be written as

$$x = (x_1, x_2, \dots, x_n) = \sum_{j=1}^n x_j e^j, \quad \text{and every}$$

$y \in \mathbb{C}^m$ is given by

(*) $$y = (y_1, \dots, y_m) = \sum_{i=1}^m y_i \phi^i, \quad \text{where } y_i = (y, \phi^i).$$

Let $A \in \mathcal{L}(\mathbb{C}^n; \mathbb{C}^m)$. Then by linearity

$$Ax = \sum_{j=1}^n A x_j e^j = \sum_{j=1}^n x_j A e^j, \quad \text{and}$$

by (*), $$Ax = \sum_{i=1}^m (Ax)_i \phi^i = \sum_{i=1}^n (Ax, \phi^i) \phi^i$$
$$= \sum_{i=1}^m \sum_{j=1}^n x_j (A e^j, \phi^i) \phi^i.$$

Thus, $$\boxed{(Ax)_i = \sum_{j=1}^n A_{ij} x_j, \quad \text{where } A_{ij} = (Ae^j, \phi^i),}$$

which is the standard "matrix times vector" formula.

The same idea works in infinite dimensions.

7.23 Defn. Let E, F be Hilbert spaces with complete orthonormal bases $\{e_n\}_{n \in \mathbb{N}}$ and $(\phi_m)_{m \in \mathbb{N}}$, respectively, and let $A \in \mathcal{L}(E; F)$. Then the matrix of A with respect to $\{e_n\}$ and $\{\phi_m\}$ is the (infinite) array $\{a_{ij}\}_{i,j=1}^\infty$ of complex (or real) numbers given by

$$\boxed{a_{ij} = (Ae_j, \phi_i)}.$$

7.24 Thm: If $\{a_{ij}\}_{i,j=1}^\infty$ is the matrix of A, then for every

$$x = \sum_{j=1}^\infty x_j e_j = \sum_{j=1}^\infty (x, e_j) e_j.$$

we have $(Ax)_i = (Ax, \phi_i) = \sum_{j=1}^\infty a_{ij} x_j$.

Thus, $$Ax = \sum_{i=1}^\infty \Big(\sum_{j=1}^\infty a_{ij} x_j \Big) \phi_i.$$

Proof: (Same as matrix case):

Every $y \in F$ is of the form
$$y = \sum_{i=1}^{\infty} y_i f_i = \sum_{i=1}^{\infty} (y, f_i) f_i .$$
In particular,
$$Ax = \sum_{i=1}^{\infty} (Ax)_i f_i = \sum_{i=1}^{\infty} (Ax, f_i) f_i .$$
On the other hand, $x = \lim_{N\to\infty} \sum_{j=1}^{N} (x, e_j) e_j$,

and by the linearity and continuity of A,
$$Ax = A\left(\lim_{N\to\infty} \sum_{j=1}^{N} (x, e_j) e_j\right)$$
$$= \lim_{N\to\infty} \sum_{j=1}^{N} (x, e_j) A e_j =$$
$$= \lim_{N\to\infty} \sum_{j=1}^{N} x_j A e_j .$$

Take the inner product with f_i to get
$$(Ax)_i = (Ax, f_i) = \lim_{N\to\infty} \sum_{j=1}^{N} x_j (Ae_j, f_i)$$
$$= \lim_{N\to\infty} \sum_{j=1}^{N} a_{ij} x_j , \quad \text{so}$$
this sum converges, and we can write it as $\sum_{j=1}^{\infty} a_{ij} x_j$. $\square$

Problem: Not every collection of numbers $\{a_{ij}\}_{i,j=1}^{\infty}$ defines an operator $E \to F$!

There are some easy necessary conditions, but no obvious (nontrivial) sufficient conditions.

Example: If $\{a_{ij}\}_{i,j=1}^{\infty}$ is diagonal, i.e. $a_{ij} = 0$ for $i \neq j$, then A is bounded if and only if $\sup_i |a_{ii}| < \infty$.

Proof: This is a home work!

Necessary conditions:

1) For every $j \in \mathbb{N}$, $Ae_j \in F$, so
$$Ae_j = \sum_{i=1}^{\infty} (Ae_j, f_i) f_i , \quad \text{and}$$
$$\|Ae_j\|_F^2 = \sum_{i=1}^{\infty} |(Ae_j, f_i)|^2 = \sum_{i=1}^{\infty} |a_{ij}|^2 .$$
Thus, $\underline{\underline{\sum_{i=1}^{\infty} |a_{ij}|^2 < \infty}}$

2) For every $i \in \mathbb{N}$, $\sum_{j=1}^{\infty} |a_{ij}|^2 < \infty$.
This follows from 1) and the following thm:

7.25 Thm. Let $\{a_{ij}\}_{i,j=1}^{\infty}$ be the matrix of $A \in \mathcal{L}(E;F)$ with respect to the orthonormal bases $\{e_n\}$ in E and $\{f_m\}$ in F. Then the matrix of the adjoint operator $A^* \in \mathcal{L}(F;E)$ with respect to the basis $\{f_m\}$ in F and the basis $\{e_n\}$ in E is
$$(A^*)_{ij} = \{\bar{a}_{ji}\}_{i,j=1}^{\infty} \quad \text{(complex conjugate transpose).}$$

Proof: $(A^* f_i, e_j) = (f_i, Ae_j) = \overline{(Ae_j, f_i)} = \overline{(A_{ji})}$.

7.26. Pseudothm Most other "reasonable" matrix results remain true. Thus,

$$\boxed{\text{operators can be reduced to } \infty\text{-dim matrices}}$$

(for example for numerical computations)