Next (N2): This is easy.

Finally (N3): $|F(x)+G(x)| \le |F(x)|+|G(x)|$ for all x, hence

$$\sup_{\substack{x\in E\\ \|x\|\le 1}} |F(x)+G(x)| \le \sup_{\substack{x\in E\\ \|x\|\le 1}} |F(x)| + |G(x)|$$

$$\le \sup_{\|x\|\le 1} |F(x)| + \sup_{\|x\|\le 1} |G(x)| = \|F\|_{E^*} + \|G\|_{E^*}.$$

This proves that $\|\cdot\|_{E^*}$ is a norm.

Completeness? Take a Cauchy-sequence in E^*. Then

$$(*) \quad \sup_{\|x\|\le 1} |F_n(x) - F_m(x)| \to 0 \quad \text{as } \begin{cases} m\to\infty \\ n\to\infty. \end{cases}$$

Therefore, for each *fixed* $x\in E$,

$$|F_n(x) - F_m(x)| \to 0 \quad \text{as } \begin{cases} m\to\infty \\ n\to\infty. \end{cases}$$

Since the field ($=\mathbb{R}$ or $=\mathbb{C}$) is complete, this converges to a limit. Call the limit $F(x)$.

We claim that the mapping $x \mapsto F(x)$ is a continuous linear functional.

Linearity of F: "easy" (just write out what it means)

Continuity of F: We use Thm 6.3: First pick n and m so large that (by $(*)$)

$$\sup_{\|x\|\le 1} |F_n(x) - F_m(x)| < \varepsilon. \quad \text{Let}$$

$m\to\infty$ to get $\boxed{\sup_{\|x\|\le 1} |F_n(x) - F(x)| < \varepsilon.}$ $(**)$ By Thm 6.3, there is some $M<\infty$ so that

$$\sup_{\|x\|\le 1} |F_n(x)| \le M. \quad \text{Therefore}$$

$$\sup_{\|x\|\le 1} |F(x)| \le \sup_{\|x\|\le 1} |F(x) - F_n(x) + F_n(x)| \le M+\varepsilon.$$

By Thm 6.3, F is continuous.

Does $F_n \to F$ in the norm of E^*, i.e., is it true that

$$\|F_n - F\|_{E^*} = \sup_{\|x\|\le 1} |F_n(x) - F(x)| \to 0 \quad \text{as } n\to\infty.$$

Yes! See $(**)$ on previous page. $\square$

Defn. We call E^* the (Banach) Dual of E.

Note: This is a very powerful method of
A) creating new Banach spaces
B) proving that some normed spaces are complete.

6.6 Thm. Let $x\in E$ and $F \in E^*$. Then

$$|F(x)| \le \|F\|_{E^*} \|x\|_E.$$

Proof Trivial if $x=0$. If $x\ne 0$, then we define $x^0 = x/\|x\|$, and $\|x^0\| = 1$, so by the definition of $\|F\|_{E^*}$,

$$\|F\|_{E^*} = \sup_{\|y\|=1} |F(y)| \ge |F(x^0)|$$

$$= |F(x/\|x\|)| = \frac{1}{\|x\|} |F(x)|, \quad \text{i.e.,}$$

$$|F(x)| \le \|F\|_{E^*} \|x\|_E. \qquad \square$$

6.7 Example $\boxed{\text{"}\ell^\infty \text{ is the dual of } \ell^1\text{"}}$.

Here ℓ^1 = sequences $\{a_n\}_{n=1}^\infty$ satisfying $\|a\| = \sum_{n=1}^\infty |a_n| < \infty$

and ℓ^∞ = sequences $\{a_n\}_{n=1}^\infty$ satisfying $\|a\|_{\ell^\infty} = \sup_{n\ge 1} |a_n|$.

More precisely: There is a unitary mapping between $(\ell^1)^*$ and ℓ^∞.

Suppose that $F \in (\ell^1)^*$. Let $e_n^k = \delta_n^k = \begin{cases} 1, & n=k \\ 0, & n\ne k \end{cases}$

Thus, $e^1 = \{1, 0, 0, \dots\}$
$e^2 = \{0, 1, 0, \dots\}$ etc.
(We know that this is an orthonormal basis in ℓ^2, but this is not important now).

Let $c_k = F(e^k)$. By linearity, if $x \in \ell^1$, and only finitely many $x_k \neq 0$, say

$$x = \{x_1, x_2, x_3, \dots, x_N, 0, 0, 0, \dots\},$$

then we can write

$$x = \sum_{k=1}^{N} x_k e^k,$$

and by linearity $F(x) = F\left(\sum_{k=1}^{N} x_k e^k\right) \Rightarrow$

(*) $$\boxed{F(x) = \sum_{k=1}^{N} x_k F(e^k) = \sum_{k=1}^{N} x_k c_k}.$$

(This looks like an inner product without the complex conjugation!)

Claim: $\{c_k\}_{k=1}^\infty = \{F(e^k)\}_{k=1}^\infty \in \ell^\infty$, and for all $x \in \ell^1$,

$$F(x) = \sum_{k=1}^\infty x_k c_k.$$

Proof: First: why does $\{c_k\} \in \ell^\infty$? Because:

(**) $$|c_k| = |F(e^k)| \le \|F\| \|e^k\| = \|F\|$$

(since $\|e^k\| = \sum_{n=1}^\infty |e^k_n| = \sum_{n=1}^\infty \delta^k_n = 1$).
Thus, $\{c_k\}_{k=1}^\infty \in \ell^\infty$.

Next: why is $F(x) = \sum_{k=1}^\infty x_k c_k$? Because:

We know this if $x_n = 0$ for $n \ge N$. If not, then we approximate x by $y_N = \sum_{k=1}^N x_k e^k$. Then,

$$\|y_N - x\| = \sum_{k=N+1}^\infty |x_k| \to 0 \text{ as } N \to \infty,$$

so $y_N \to x$ in ℓ^1.

By (*), $F(y_N) = \sum_{k=1}^{N} x_k c_k$.

Let $N \to \infty$, and use the continuity of F to get

$$F(x) = \lim_{N\to\infty} \sum_{k=1}^{N} x_k c_k = \sum_{k=1}^{\infty} x_k c_k.$$

(This sum converges actually absolutely.

This proves: To every $F \in (\ell^1)^*$ there is a sequence $\{c_n\}_{n=1}^\infty \in \ell^\infty$ such that

(***) $$F(x) = \sum_{n=1}^\infty c_n x_n.$$

Moreover, $\|c\|_{\ell^\infty} \le \|F\|_{(\ell^1)^*}$ (see **).

Conversely, take some $c \in \ell^\infty$, and define $F(x)$ by (***). Then it is easy to show that $F \in (\ell^1)^*$, and that $\|F\|_{(\ell^1)^*} \le \|c\|_{\ell^\infty}$.

Thus, the mapping $F \Leftrightarrow c$ described above is a unitary map of $(\ell^1)^*$ onto ℓ^∞:

Every $F \in (\ell^1)^*$ is of the form (***), and every $c \in \ell^\infty$ induces an $F \in (\ell^1)^*$ by the formula (***). Moreover, $\|F\|_{(\ell^1)^*} = \|c\|_{\ell^\infty}$.

Example: A very similar computation shows that "the dual of c_0 is ℓ^1". We repeat the preceding computation, taking $x \in c_0$, $F \in (c_0)^*$, define $c_k = F(e^k)$, conclude that $c \in \ell^1$, and find that (***) defines a unitary mapping between $(c^0)^*$ and ℓ^1.

Thus: "The dual of c_0 is ℓ^1, and the dual of ℓ^1 is ℓ^∞".
(In particular, by thm 6.5, ℓ^1 and ℓ^∞ are complete).

VI.5 The dual of a Hilbert space

In the case of a Hilbert space Theorem 6.5 simplifies as follows:

6.8 Riesz' Representation theorem. Let H be a Hilbert space.

A) For each $y \in H$, the function

$$(*) \quad F(x) = (x, y), \quad x \in H.$$

is a continuous linear functional on H (i.e., $F \in H^*$; see Thm 6.5).

B) Conversely, to every $F \in H^*$ there correspond a unique $y \in H$ so that F is given by $(*)$.

C) In both cases, $\|F\|_{H^*} = \|y\|_H$.

Proof. A) is example (iv) on p. 40 of the notes.

C) By Cauchy-Schwartz, $|(x,y)| \le \|x\|\,\|y\|$. Taking $\|x\| \le 1$ we get $\|F\|_{H^*} \le \|y\|$. On the other hand

$$\|F\|_{H^*} = \sup_{\|x\| \le 1} |(x,y)| \ge \left|\left(\frac{y}{\|y\|}, y\right)\right| \quad \left(\text{take } x = y/\|y\|\right)$$

$$= \frac{1}{\|y\|}(y,y) = \frac{1}{\|y\|}\|y\|^2 = \|y\|^2.$$

Thus, $\|F\|_{H^*} = \|y\|$ (the computation above used the assumption $y \ne 0$, the case $y = 0$ is trivial).

B) If F is the zero functional, then we take $y = 0$. If not, then $F(v) \ne 0$ for some $v \ne 0$. The set $M = \{x \in H \mid F(x) = 0\}$ is a closed subspace of H (easy to show).

This is not the whole space since $v \notin M$. Therefore $M^\perp \ne \{0\}$ (see Corollary 4.30). Pick some $w \in M^\perp$, $w \ne 0$, and define

$$z = w / F(w) \quad (\text{note that } F(w) \ne 0 \text{ since } w \in M^\perp,\ w \ne 0)$$

Then $F(z) = F\left(\frac{1}{F(w)} w\right) = \frac{1}{F(w)} F(w) = 1$, so $\boxed{F(z) = 1}$.

Claim The vector $y = z/\|z\|^2$ can be used in $(*)$.

Proof. Every $x \in H$ can be split (uniquely) into $x = x_1 + x_2$ where $x_1 \in M$ and $x_2 \in M^\perp$. Now $x = x - F(x)z + F(x)z$, where

$F(x - F(x)z) = F(x) - F(x)F(1) = 0$, so

$x_1 = x - F(x)z \in M$ and $x_2 = F(x)z \in M^\perp$.

Thus, $x_1 \perp z$, and

$(x, z) = (x_2, z) = (F(x)z, z) = F(x)\|z\|^2$.

This means that, taking $y = z/\|z\|^2$, we get $\boxed{F(x) = (x,y)}$, as required by $(*)$. $\square$

Warning: The mapping $F(x) \leftrightarrow y$ is is not an unitary operator from H^* to H. It does satisfy most of the conditions of a unitary operator: It preserves norms, i.e., $\|F\|_{H^*} = \|y\|_H$, it maps H^* one-to-one onto H, but it is not linear but conjugate linear:

If $F \leftrightarrow y$
$G \leftrightarrow z$

then for complex λ and μ

$$(\lambda F + \mu G)(x) = \lambda F(x) + \mu G(x)$$

$$= \lambda (x,y) + \mu (x, z) = (x, \bar\lambda y + \bar\mu z), \text{ so}$$

$$\boxed{(\lambda F + \mu G) \leftrightarrow \bar\lambda y + \bar\mu z}$$

Thus, there is a conjugate linear norm preserving map between H^* and H.

VII Linear Operators

VII.1 Introduction

See the nice intro in the book.

VII.2 Definitions and examples

7.1 Defn. A) Let E, F be vector spaces over k (where $k=\mathbb{R}$ or $k=\mathbb{C}$). A linear operator from E to F is a mapping $T: E\to F$ satisfying
$$T(\lambda x+\mu y) = \lambda T(x)+\mu T(y), \quad x,y\in E,\ \lambda,\mu\in k.$$
When $E=F$, we call T an operator on E ("på").
When $F=k$, we call T a linear functional on E.

B) If E and F are normed spaces, then T is bounded if there exists a constant $M<\infty$ such that
$$\|Tx\|_F \le M\|x\|_E \qquad \forall x\in E.$$

C) The norm of a bounded operator T is given by
$$\|T\|_{\mathcal{L}(E;F)} = \sup_{\|x\|_E\le 1} \|Tx\|_F .$$
("kärna")

D) The kernel or nullspace of T is the subspace $\{x\in E \mid T(x)=0\}$ of E. Denoted by $Ker(T)$.

E) The range ("värdemängd") of T is the subspace $\{y\in F \mid y=T(x) \text{ for some } x\in E\}$ of F.

Note: Most of these are the same definitions as we use for linear functionals, but the range of a linear functional is trivial: It is either k (if $T\ne 0$) or $\{0\}$ (if $T=0$).

Note: It is easy to see that for all $x\in E$,
$$\|Tx\|_F \le \|T\|\,\|x\|$$
(write $x=\|x\|z$ where $z=x/\|x\|$ and use the definition of $\|T\|$). Conversely, $\|T\|$ is the smallest number M for which

(*) $$\|Tx\|_F \le M\|x\|.$$

(since this implies that $\|Tx\|\le M$ if $\|x\|\le 1$).

7.1a Lemma The kernel of a bounded linear operator is a closed subspace.

Proof. Let $x_n\in Ker(T)$, and $x_n\to x$ in E. Then
$$\|Tx\|_F = \|Tx-Tx_n+Tx_n\|_F \le \|Tx-Tx_n\|_F + \underbrace{\|Tx_n\|_F}_{=0}$$
$$= \|T(x-x_n)\|_F \le M\|x-x_n\|_E .$$
Let $n\to\infty$. Then $\|x-x_n\|_E\to 0$, and so $\|Tx\|=0$, i.e., $x\in Ker(T)$. $\square$

Varning The range of a bounded linear operator is also a subspace, but it need not be closed.

7.2a Ex. Let $f\in C[0,1]$, $E=F=L^2(0,1)$, and define the multiplication operator M_f by
$$M_f(g) = f(t)g(t), \quad 0\le t\le 1.$$
This is a linear operator on $E=L^2(0,1)$. We use the standard norm in L^2, i.e.,
$$\|g\|_2 = \left(\int_0^1 |g(s)|\,ds\right)^{1/2}.$$

Then, for all $g \in L^2(0,1)$,

$$\|M_f(g)\|_2^2 = \int_0^1 |\phi(t)g(t)|^2\,dt$$
$$\le \int_0^1 \left(\sup_{0\le s\le 1} |\phi(s)|\right)^2 |g(t)|^2\,dt$$
$$= \left(\sup_{0\le s\le 1} |\phi(s)|\right)^2 \int_0^1 |g(t)|^2\,dt \text{, so}$$

$$\|M_f(g)\|_2 \le \sup_{0\le s\le 1} |\phi(s)|\ \|g\|_2.$$

This means that M_f is bounded, and that $\|M_f\| \le \sup_{0\le s\le 1} \phi(s) = \|f\|_{max}$.

Actually, we have $\|M_f\| = \|f\|_{max}$. Proof of this:

Choose $s_0 \in [0,1]$ so that

$$|\phi(s_0)| = \max_{0\le s\le 1} |\phi(s)|.$$

Let $\varepsilon > 0$. Since ϕ is continuous, we can find $\delta > 0$ so that $|\phi(s)| \ge |\phi(s_0)| - \varepsilon$ for $|s - s_0| < \delta$. Define

$$I = [s_0-\delta, s_0+\delta] \cap [0,1],$$

$$g(s) = \chi_I(s) = \begin{cases} 1, & s \in I, \\ 0, & s \notin I. \end{cases}$$

Denote $|I|$ = length of I. Then

$$\|M_f(g)\|_2^2 = \int_0^1 |\phi(t)g(t)|^2\,dt = \int_I |\phi(t)|^2\,dt$$
$$\ge (|\phi(s_0)| - \varepsilon)^2 |I|, \text{ and}$$
$$\|g\|_2^2 = \int_0^1 |g(t)|^2\,dt = \int_I dt = |I|, \text{ so}$$
$$\|M_f(g)\|_2^2 \ge (|\phi(s_0)| - \varepsilon)^2 \|g\|_2^2.$$

Thus, the constant M in ⓕ (on page 54) must be at least $|\phi(s_0)| - \varepsilon$, and ε is arbitrary $\Rightarrow$ $\|M_f\| \ge |f(s_0)| = \|\phi\|_{max}$. ∎

7.26 Ex. Let $E = L^2(a,b)$, $F = L^2(c,d)$, and let k be continuous $[a,b]\times[c,d] \to \mathbb{C}$ (or $\mathbb{R}$). We define the integral operator T by

$$(Tx)(t) = \int_a^b k(t,s)\,x(s)\,ds, \quad c<t<d.$$

Clearly T is linear. Is it bounded? Yes, because for each t, by Cauchy-Schwartz,

$$|(Tx)(t)|^2 \le \int_a^b |k(t,s)|^2\,ds \int_a^b |x(s)|^2\,ds,$$

and by integrating this inequality over t we get

$$\|Tx\|^2 = \int_c^d |(Tx)(t)|^2\,dt$$
$$\le \left(\int_{t=c}^d \int_{s=a}^b |k(t,s)|^2\,ds\,dt\right) \|x\|_2^2.$$

This shows that T is bounded, and that

$$\|T\| \le \left(\iint |k(t,s)|^2\,ds\,dt\right)^{1/2}.$$

(We do not always have equality here. This is more difficult to show?).

7.2c Ex. An unbounded differentiation operator (all differentiation operators are unbounded in some sense). If we try to define an unbounded operator T on some Hilbert space H, then we cannot let the domain of this operator (the set of $x \in H$ for which Tx is defined) be the whole space. In this case we take the space H to be $L^2(-1,1)$, and the domain $\mathcal{D}$ to be

$$\mathcal{D} = \{ \phi \in C(-1,1) \mid \phi' \in L^2(-1,1) \}.$$

(ϕ is differentiable almost everywhere and $\phi' \in L^2$).

Let $T: \mathcal{D} \to L^2(-1,1)$ be

$$T\phi = \phi'.$$

Then T is linear from $\mathcal{D}$ to $L^2(-1,1)$, but there is no constant $M < \infty$ such that

$$\|T\phi\|_{L^2} \le M \|\phi\|_{L^2}, \quad \phi \in \mathcal{D}$$

(note: same norm in both places). Thus, it is impossible to extend the domain of T to $L^2(-1,1)$ so that it become a bounded operator on $L^2(-1,1)$.

7.2d Ex. Take $E = F = \ell^2$, and define the right (or outgoing) shift operator S:

$$S(x_1, x_2, x_3, x_4, \dots) = (0, x_1, x_2, x_3, \dots)$$

(it moves all coefficients one step to the right). It is easy to see that this is a bounded linear operator, which is even isometric, i.e., $\|Sx\| = \|x\|$ for all $x \in \ell^2$. Therefore $\|S\| = 1$. However, it is not onto, so it is not unitary. (nothing is mapped into $(1,0,0,0,0,\dots)$).

There is also a left (or incoming) shift which we denote by S^*:

$$S^*(x_1, x_2, \dots) = (x_2, x_3, x_4, \dots) \quad \leftarrow x_1 \text{ is missing}$$

This operator is bounded, and $\|S^*\| = 1$. It is not isometric, since it maps $(1,0,0,0,\dots)$ into zero.

Note: Examples 7.2a–c are very important in physics. For example, using the operator $T\phi = \phi'$ we can write a differential equation

$$x'(t) = \phi(t, x(t))$$

in the form $Tx = F(x)$ (right hand side $= F(x)$) and try to rewrite it in the form

$$x = T^{-1}F(x),$$

which is some sort of integral equation; and T^{-1} is an integral operator of the type described in Ex. 7.2b.

Example 7.2d is very important in Fourier analysis, harmonic analysis, and time series.

VII 3 Basic Properties

7.4 Thm Let E, F be normed spaces, and let $T: E \to F$ be a linear operator. Then t.f.c.a.e.

(i) T is uniformly continuous
(ii) T is continuous at the point zero
(iii) T is bounded
(iv) $\|T\| < \infty$.

Proof: Identical to the proof of Thm 6.3. Replace $|\cdot|$ by $\|\cdot\|$. $\square$

7.5 Thm. Let $\mathcal{L}(E;F)$ consist of all bounded linear operators $E \to F$. This is a normed space with the norm $\|\cdot\|$ defined in Defn. 7.1, and it is a Banach space if F is a Banach space.

Proof. Identical to the proof of Thm. 6.5.

Note: The completeness of F is important. It does not matter if E is complete.

7.5a Defn: Let $A: E \to F$ and $B: F \to G$. Then we denote the operator

$$x \to B(A(x)) \quad \text{by } \underline{BA}$$

and we call it the composition of B and A.

7.6 Thm: Let $A \in \mathcal{L}(E;F)$ and $B \in \mathcal{L}(F;G)$. Then $BA \in \mathcal{L}(E;G)$, and

$$\|BA\| \leq \|B\|\,\|A\|.$$

Proof easy (see book).

7.7 Coroll. $\|A^n\| \leq \|A\|^n$.

Proof by induction from Thm 7.6.

XII 4 Projection Operators

(This section comes much later in the book.)

Defn: An operator $P \in \mathcal{L}(E)$ (where E is a normed space) is a projection operator if $P^2 = P$. If E is a inner product space, then we say that P is orthogonal if $\mathcal{R}(P) \perp \mathcal{N}(P)$ (i.e., every $y \in \mathcal{R}(P)$ is orthogonal to every $x \in \mathcal{N}(P)$).

Interpretation: After the first application of P to a vector x we get a vector which does no longer change if we ~~later~~ repeat the same operation.

Lemma: For every projection P, both $\mathcal{N}(P)$ and $\mathcal{R}(P)$ are closed, and E is the direct sum of $\mathcal{N}(P)$ and $\mathcal{R}(P)$. This is an orthogonal direct sum if P is orthogonal. Moreover, also the operator $Q = I - P$ is a projection, and

$$\mathcal{N}(Q) = \mathcal{R}(P)$$
$$\mathcal{R}(Q) = \mathcal{N}(P)$$

$Q + P = I \Leftrightarrow P$ and Q are complementary projections

Proof: i) Q is a projection, because

$$Q^2 = (I-P)^2 = I - P - P + P^2 = I - P = Q$$

ii) We claim that $\boxed{x \in \mathcal{R}(P) \Leftrightarrow Px = x}$ and that $x \in \mathcal{R}(Q) \Leftrightarrow Qx = x$.

Proof: Obviously, if $x = Px$ then $x \in \mathcal{R}(P)$. Conversely, if $x \in \mathcal{R}(P)$, then $x = Py$ for some $y \in E$, so

$Px = P(Py) = P^2 y = Py = x$

iv) By iii), $x \in R(P) \Leftrightarrow x = Px$

$\Leftrightarrow (I-P)x = 0 \Leftrightarrow x \in N(Q)$. Thus $R(P) = N(Q)$, and similarly, $N(Q) = R(P)$.

v) We know that for all operator $P \in \mathcal{L}(E)$ the space $N(P)$ is closed. Therefor $R(Q) = N(P)$ is closed. In the same way $R(P) = N(Q)$ is closed.

vi) Every $x \in E$ can be split into

$x = Px + (I - Px) = Px + Qx$, where

$Px \in R(P)$, and $Qx \in R(Q) = N(P)$.

Furthermore, if $x \in R(P) \cap N(P)$, then $x \in N(Q) \cap N(P)$, so both $Px = 0$ and $Qx = 0$, hence $x = Px + Qx = 0$.

Thus, $E = N(P) \oplus R(P)$.

This sum is obviously orthogonal if P is an orthogonal projection. □

VII. 5 The Inverse operator.

Notation: I_E is the identity operator in E. It maps every $x \in E$ into itself.

7.8 Defn. An operator $A \in \mathcal{L}(E;F)$ is invertible if there exists an operator $B \in \mathcal{L}(F;E)$ such that

"invertierbar" $AB = I_F$ and $BA = I_E$

(i.e., $BAx = x$ for all $x \in E$, $ABy = y$ — " — $y \in F$).

We call B the inverse of A, and denote it by $B = A^{-1}$.

Is this possible? Could there exist more than one inverse? Then A^{-1} is not uniquely determined?

Lemma 7.8a. Suppose that $A \in \mathcal{L}(E;F)$, $B \in \mathcal{L}(F;E)$, $C \in \mathcal{L}(F;E)$, and that

$$BA = I_E, \quad AC = I_F.$$

Then $B = C$, and A is invertible.

In other words, if A has both a left-inverse B and a right-inverse C, then A is invertible.

Proof: $C = I_E C = (BA)C = B(AC) = B I_F = B$. □

Note: In Def 7.8 we require $B = A^{-1}$ to be bounded. It is not enough that B is a linear operator. However, B cannot be unbounded if E and F are Banach spaces. (proof later maybe, this is a difficult proof).