4.15 Thm. Let $\{e_n\}$ be an orthonormal sequence in a Hilbert space H. Then t.f.c.a.e. (= "the following conditions are equivalent"):

(1) $\{e_n\}_{n\in\mathbb{N}}$ is complete

(2) $\overline{[\{e_n\}_{n\in\mathbb{N}}]} = H$ (the closed linear span is H)

(3) $x = \sum_{n=1}^{\infty} (x,e_n)e_n$ for all $x \in H$

(4) $\|x\|^2 = \sum_{n=1}^{\infty} |(x,e_n)|^2$ — " —

(5) Only the zero vector is orthogonal to all the vectors e_n.

Proof: (1) ⟺ (5): This is Defn. 4.13.

(1) ⟹ (3): Contained in Thm 4.14

(3) ⟹ (4): We proved this as a part of Thm 4.14.

(4) ⟹ (5): Obvious ($x \neq 0 \Rightarrow (x,e_n) \neq 0$ for at least one n).

(1) ⟹ (2): Follows from Thm 4.14 (since $x_n \in$ the linear span of $\{e_n\}$, an $x_n \to x$)

(2) ⟹ (5): Suppose that (5) does not hold. Then $\exists\, x \in H$ so that $x \perp e_n$ for all n, i.e., $(x,e_n) = 0$. We claim that this vector does not belong to the closed linear hull of $\{e_n\}$.
Proof of this: If $y_k \in [\{e_n\}]$, then for some constants λ_n and some finite

$$y_k = \sum_{j=1}^{n_k} \lambda_j e_j, \quad \text{so}$$

$$(y_k, x) = \sum_{j=1}^{n_k} \lambda_j (e_j, x) = 0, \quad \text{and } y_k \perp x.$$

Thus, $\|y_k - x\|^2 = \|y_k\|^2 + \|x\|^2 \geq \|x\|^2 > 0$, so it is impossible to pick a sequence y_k from $[\{e_k\}]$ which converge to x. Thus x does not belong to $\overline{[\{e_n\}]}$, hence $\overline{[\{e_n\}]} \neq H$. ▽

IV.5 Isometrically isomorphic spaces

4.16 Defn. A Hilbert space is separable iff it has a orthonormal basis with countably many (or finitely many) basis vectors.

(Note: This definition is different but equivalent to the usual one: separable ⟺ there is a dense subset with countably many points.)

4.17. Defn. Let H and K be two normed spaces.
i) An operator $A: H \to K$ is linear if
$$A(\lambda x + \mu y) = \lambda Ax + \mu Ay \quad \text{for all } x,y \in H, \ \lambda,\mu \in \mathbb{C}.$$
ii) This operator is isometric if, in addition
$$\|Ax\|_K = \|x\|_H \quad \text{(i.e., it preserves norms).}$$
iii) It is unitary if, in addition, it is onto, i.e., every $y \in K$ is of the form $y = Ax$ for some $x \in H$.

4.18. Thm. i) Every unitary operator U is invertible, and U^{-1} is also unitary.

ii) If U is an isometric operator from one inner product space to another, then it also preserves inner products, i.e.,

$$(Ux, Uy)_K = (x,y)_H \quad \forall x,y \in H.$$

Proof. i) "Straightforward" (ii) and iii) in Defn 4.17 guarantee invertibility; if a linear operator is invertible then the inverse is linear; ii) shows that distances are preserved).

ii) Use the polarization identity.

4.17b Defn. Two normed spaces H and K (or inner product spaces, or Hilbert, or Banach) are isometrically isomorphic if there is a unitary operator mapping H onto K.

4.19 Thm (The "great simplification thm"): Every separable Hilbert space is either

i) isomorfic to $\mathbb{C}^n$

(this is true whenever H is finite-dimensional), or

ii) isomorphich to l^2

(this is true whenever H is infinite-dimensional).

Thus, there "is one and only one separable infinite-dimensional Hilbert space" in the following sense: If H is an arbitrary such space, then we can use a unitary mapping U from H into l^2 to replace vectors in H by vectors in l^2, then we do all computations in l^2, and finally we map the result back into H.

"Everything which is true in this H is true in l^2, and conversely".

Proof: i) In the finite-dimensional case H has a finite orthonormal basis $\{e_1, e_2, e_3 \dots, e_n\}$. For each $x \in H$ we define

$$Ux = (\alpha_1, \alpha_2, \dots, \alpha_n)$$ where

$\alpha_k = (x, e_k)$ is the uth Fourier coefficient. This is a unitary mapping from H to $\mathbb{C}^n$.

ii) In the infinite-dimensional case H has an infinite orthonormal basis $\{e_n\}_{n=1}^{\infty}$. We map every $x \in H$ onto the sequence of Fourier coefficients $\{\alpha_n\}_{n=1}^{\infty}$, where $\alpha_n = (x, e_n)$.

In both cases these mappings are linear, and by Thm 4.15 they preserve distances. The only slight problem is that these mappings are onto. But this follows from Thm 4.11: If $\{\lambda_n\} \in l^2$, then $\sum_{n=1}^{\infty} \lambda_n e_n$ converges to some $x \in H$, and $(x, e_n) = \lambda_n$. □

IV.6 Orthogonal Complements

4.20 Defn. Let E be an arbitrary subset of the inner product space H. The orthogonal complement $E^\perp$ is given by

$$\{x \in H \mid x \perp y \text{ for all } y \in E\}$$
$$= \{x \in H \mid (x, y) = 0 \text{ for all } y \in E\}.$$

Alternative notation: When E is a closed (linear) subspace of H we sometimes write $E^\perp = H \ominus E$. (but not in other cases).

4.22 Thm: For an arbitrary $E \subset H$, the set $E^\perp$ is a closed subspace of H.

Proof "easy" (homework??)

4.23 Lemma Let M be a subspace of a unitary space H. Then

$$M^\perp = \{x \in H \mid \|x - y\| \geq \|x\| \text{ for all } y \in M\}.$$

This lemma is needed in the proof of the next theorem. It says that $x \in M^\perp \Leftrightarrow$ the distance from x to M is $\|x\|$. (The point in M which lies closest to x is the origin; every other point lies further away.)

Proof. i) Assume that $x \in M^\perp$, i.e., $(x, y) = 0$ for all $y \in M$. Then also $(x, -y) = 0$ for all y, and by Pythagoras theorem

$$\|x - y\|^2 = \|x\|^2 + \|-y\|^2 \geq \|x\|^2.$$

ii) Assume that $x \notin M^\perp$, i.e., $(x, y) \neq 0$ for some $y \in M$. Then we can get a point in M which is closer to x than the origin by projecting x onto y as follows: Note that $y \neq 0$. Put

$$z = (x, y^0)\, y^0 \quad \text{where } y^0 = \frac{y}{\|y\|},$$

i.e. $z = (x, y)\, y / \|y\|^2$. Then $z \neq 0$ and

$$\underbrace{(x - z, y) = (x, y) - (x, y) \underbrace{\left(\frac{y}{\|y\|^2}, y\right)}_{=1}}_{=0}$$

so $x - z \perp y$, hence $x - z \perp z$.

By Pythagoras theorem,

$$\|x\|^2 = \|x - z\|^2 + \|z\|^2 > \|x - z\|^2, \text{ so}$$

there is at least one $z \in M$ for which $\|x - z\|^2 < \|x\|^2$. □

(Note: More complicated proof in the book)

4.24 Thm Let M be a closed subspace of a Hilbert space H. Then it is possible to split an arbitrary vector $x \in H$ into

$$\boxed{x = y + z, \quad \text{where } y \in M,\ z \in M^\perp.}$$

This splitting is unique, and y is the point in M which is closest to x.

Proof. By Thm 1.1, there is a unique point $y \in M$ which is closest to x, i.e.,

$$(*) \quad \|x - y\| = \inf_{w \in M} \|x - w\| = \min_{w \in M} \|x - w\|.$$

Let $z = x - y$. For every $v \in M$ we have from $(*)$

$$\|z - v\| = \|x - (y + v)\| \geq \|x - y\| = \|z\|$$

since $y + v \in M$. By Lemma 7.3, $z \in M^\perp$. This shows that $x = y + z$ with $y \in M$ and $z \in M^\perp$.

Uniqueness: If $x = y + z$ with $y \in M$ and $z \in M^\perp$, then by Lemma 6.3,

$$\|z - w\|^2 \geq \|z\| \text{ for all } w \in M.$$

Thus, $\|z - w\|^2 \geq \|x - y\|^2$ for all $w \in M$, so y is the closest point in M to x, and this point is unique. □

4.25a Defn. We call the vector y in Thm 6.4 the projection of x onto M, and z the projection of x onto $M^\perp$.

4.25b Coroll. If M is a closed subspace of a Hilbert space H, and if $M \neq H$, then $M^\perp \neq \{0\}$ (i.e., $M^\perp$ contains at least one vector $\neq 0$).

4.25c Coroll. i) For all subsets M of H,

$$M \subset (M^\perp)^\perp$$

ii) We have $M = (M^\perp)^\perp$ if and only if M is a closed subspace of H.

Proof. i) is trivial (why?)

ii) "$\Rightarrow$" If $M = (M^\perp)^\perp = (E)^\perp$, where $E = M^\perp$, then by Thm. 4.22, M is a closed subspace.

"$\Leftarrow$" Assume that M is a closed subspace. Take $x \in (M^\perp)^\perp$. Use Thm. 4.24 to find unique vectors $y \in M$ and $z \in M^\perp$ so that $x = y + z$. As $x \in (M^\perp)^\perp$ we have $x \perp z$, i.e., $(x, z) = 0$. Thus

$$0 = (x, z) = (y + z, z) = \overbrace{(y, z)}^{=0 \text{ since } z \perp y} + (z, z) = \|z\|^2.$$

Thus $z = 0$ and so $x = y \in M$. This shows that $(M^\perp)^\perp \subset M$, and hence $(M^\perp)^\perp = M$. □

(Alternatively, use Thm. 4.28 below)

4.26 Defn. If M and N are closed subspaces of V, then V is the *direct sum* of M and N, denoted $V = M \oplus N$, if

i) $M \cap N = \{0\}$
ii) $V = M + N$, i.e., every $x \in V$ is of the form $x = m+n$ where $m \in M$, $n \in N$.

If, moreover V is an inner product space and

iii)' $(x,y) = 0 \quad \forall x \in M, \ \forall y \in N$,

then V is the *orthogonal direct sum* of M and N.

Warning: Some book denote the $\boxed{\text{orthogonal}}$ direct sum by $x \oplus y$!

Theorem 4.24 reformulated:

4.27 Cor. If M is a closed (linear) subspace of H, then $H = M \oplus M^\perp$, and this sum is orthogonal.

4.28 Thm. For every subset $A \subset H$,

$$(A^\perp)^\perp = \overline{[A]}.$$

Proof: We know that $\overline{[A]}$ is the smallest closed subspace containing A, and $(A^\perp)^\perp$ is one such subspace (see Thm 4.22 and Cor. 4.25b). Thus $\overline{[A]} \subset (A^\perp)^\perp$. If $\overline{[A]} \neq (A^\perp)^\perp$, then by Cor 4.25b, there is some nonzero $x \in (A^\perp)^\perp$ which is orthogonal to $\overline{[A]}$. The latter condition implies $x \in A^\perp$. Thus, $x \in A^\perp$ and $x \in (A^\perp)^\perp \Rightarrow x = 0$. This is impossible, hence $\overline{[A]} = (A^\perp)^\perp$. □

4.29 Defn. A subset A of H is *dense* (total) in H iff $\overline{A} = H$.

4.30 Corollary. Let $A \subset H$. Then

(i) $[A]$ is dense in H $\iff$

(ii) $A^\perp = \{0\}$.

Proof: We have $\{0\}^\perp = H$, hence (ii) is equivalent to $(A^\perp)^\perp = H$, and by Thm 2.8, this means that $\overline{[A]} = H$, i.e., $[A]$ is dense in H. □

Note: We call A *total* or *fundamental* if (i) and (ii) of Coroll. 4.30 hold.

V Classical Fourier Series

V.1 Basic definitions

We get the "classical" Fourier series theory by using the orthonormal sequence

$$\{e_n\}_{n=-\infty}^{\infty} = \left\{\frac{1}{\sqrt{b-a}} e^{\frac{2\pi i n x}{b-a}}\right\}$$

in $L^2(a,b)$. Some fairly usual choices of a and b are $(a,b) = (0,1)$ or $(a,b) = (-\pi,\pi)$, or $(a,b) = (0,2\pi)$. In Analysis II we used $(a,b) = (0,1)$. Here we take $(a,b) = (-\pi,\pi)$, as in the book.

5.1 Thm. The sequence $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ is a complete orthonormal sequence in $L^2(-\pi,\pi)$.

Proof (outline). Easy: $\{e_n\}_{n=-\infty}^{\infty}$ is orthonormal. Difficult: This sequence is complete. Since we cannot present a complete proof we simply skip it.

Completeness means that if $(\phi, e_n) = 0$ for all $n \in \mathbb{Z}$, then $\phi(x) = 0$ "almost everywhere". (More details found in the book, and some more comments will be given later).

By combining Thm 5.1 with Thm. 4.15 we get the following result:

5.4 Coroll. Every $\phi \in L^2(-\pi,\pi)$ can be expanded in a Fourier series

$$(*) \quad \phi = \sum_{n=-\infty}^{\infty} \hat{\phi}(n) e_n,$$

where $\hat{\phi}(n) = (\phi, e_n)$ is the n-th Fourier coefficient of ϕ, and $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$.

The convergence in $(*)$ is in the sense of unconditional L^2-convergence, i.e.,

$$\lim_{\substack{N\to\infty \\ M\to\infty}} \left\| \phi - \sum_{n=-N}^{M} \hat{\phi}(n) e_n \right\| = 0.$$

Moreover, $\{\hat{\phi}(n)\}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z})$, and

$$\|\phi\|_{L^2(-\pi,\pi)} = \|\hat{\phi}\|_{\ell^2(\mathbb{Z})}, \quad \text{i.e.,}$$

$$\int_{-\pi}^{\pi} |\phi(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\hat{\phi}(n)|^2.$$

Proof: This follows from Thm 4.15 and 5.1. □

5.6 Parsevals theorem Let $\phi \in L^2(-\pi,\pi)$ and $g \in L^2(-\pi,\pi)$ have the Fourier series

$$\hat{\phi}(n) = (\phi, e) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \phi(x) e^{-inx} dx$$

$$\hat{g}(n) = (g, e) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} g(x) e^{-inx} dx.$$

Then

$$(\phi, g)_{L^2(-\pi,\pi)} = (\hat{\phi}, \hat{g})_{\ell^2(\mathbb{Z})}, \quad \text{i.e.,}$$

$$\int_{-\pi}^{\pi} \phi(x) \overline{g(x)} dx = \sum_{n=-\infty}^{\infty} \hat{\phi}(n) \overline{\hat{g}(n)}.$$

Thus, the "finite Fourier transform", i.e., the mapping $\phi \mapsto \hat{\phi}$, is an unitary operator mapping $L^2(-\pi,\pi)$ onto $\ell^2(\mathbb{Z})$.

Proof: First part: Use Thm 4.18 ii) and Cor. 5.4. Second part: See Thm 4.19 and its proof. □

Note: Convergence in L^2 does not imply pointwise convergence. To save some time we skip the question on pointwise convergence in this course. See "Fourier transform" or the book.

Note: The same result is true for every complete orthonormal sequence (= orthonormal basis).

VI Dual Spaces

VI.1 Introduction

See the introduction in the book!
Related to "transposed matrix" or "adjoint matrix".

VI.2 Linear functionals

6.1 Defn. Let E be either a real or complex vector space and let $k=\mathbb{R}$ or $k=\mathbb{C}$ depending on whether E is real or complex.
A *linear functional* on E is a function $\phi: E \to k$, satisfying

(*) $\phi(\lambda x+\mu y) = \lambda \phi(x)+\mu \phi(y)$

for all $\lambda, \mu \in k$, $x, y \in E$.

Note: ϕ is real-valued if E is a real space
ϕ is complex-valued if E is a complex space.

Examples: (i) $E=\mathbb{C}^n$, $k=\mathbb{C}$, and

$$F(x_1, x_2, \dots, x_n) = c_1 x_1 + c_2 x_2 + \dots + c_n x_n,$$

where $c_1, c_2, \dots, c_n$ are fixed numbers in $\mathbb{C}$.

(ii) $E=\ell^1$, $k=\mathbb{C}$, and $\{c_n\}_{n=1}^\infty \in \ell^\infty$,
i.e., $\|c\|_{\ell^\infty} = \sup_{n\ge 1} |c_n| < \infty$. For each $x \in \ell^1$ we define $F(x) = \sum_{n=1}^\infty c_n x_n$.

This sum converges absolutely since, for all n, $|c_n| \le \|c\|_{\ell^\infty}$, and $\sum_{n=1}^\infty |x_n| < \infty$ (since $x \in \ell^1$), hence

$$\sum_{n=1}^\infty |c_n x_n| \le \sum_{n=1}^\infty \|c\|_{\ell^\infty} |x_n| = \|c\|_{\ell^\infty} \sum_{n=1}^\infty |x_n| = \|c\|_{\ell^\infty} \|x\|_{\ell^1}.$$

It is easy to show that this function is also *linear*, i.e., it satisfies (*) in Defn. 6.1.

(iii) $E = C[0,1]$, $k=\mathbb{R}$ (real-valued continuous functions).
Take some (bounded) Riemann-integrable (or Lebesgue integrable) function c, and for all $\phi \in E$, define

$$F(\phi) = \int_0^1 c(x)\phi(x)\,dx.$$

This integral exists (since $\phi \in C[0,1]$), F maps $C[0,1]$ into $k=\mathbb{R}$, and it is linear.

(iv) E is an inner product space, $y \in E$, and we define

$$F(x) = (x, y).$$

Then F maps $E \to k$ (where $k=\mathbb{R}$ or $k=\mathbb{C}$), and F is linear.

Comment: The last example will be the most important, and it is "the only possible one" in the Hilbert space case.

VI.3 Continuous linear functionals

6.16 Defn: Recall that $\phi: E \to k$ (E a normed space) continuous at the point $y \in E$ iff

$\forall \varepsilon > 0$ there is a $\delta > 0$ such that $|\phi(x) - \phi(y)| < \varepsilon$ for all x satisfying $\|x-y\| < \delta$ (δ may depend on y)

It is *uniformly* continuous if it is continuous at every point $y \in E$, and δ does not depend on y:

$\forall \varepsilon > 0$ there is a $\delta > 0$ such that $|\phi(x) - \phi(y)| < \varepsilon$ for all $x, y \in E$ with $\|x-y\| < \delta$

6.3 Thm. Let E be a normed space with norm $\|\cdot\|$, and let $F: E \to K$ be a linear functional. Then t.f.c.a.e.

(i) F is uniformly continuous
(ii) F is continuous at the point zero
(iii) $|F(x)| \le M\|x\|$ for some $M < \infty$ and all $x \in E$
(iv) $\sup\{|F(x)| \mid x \in E, \|x\| \le 1\} < \infty$.

Proof. $\boxed{(i) \Rightarrow (ii)}$: Trivial

$\boxed{(ii) \Rightarrow (iii)}$: Assume (ii). Take $\varepsilon = 1$. Then there is some $\delta > 0$ such that $|F(y)| < 1$ for all $y \in E$ with $\|y\| < \delta$. Let $x \in E$ be arbitrary. Define $y = \frac{\delta}{2\|x\|} x$. Then $\|y\| = \|\frac{\delta}{2\|x\|} x\| = \frac{\delta}{2\|x\|}\|x\| = \delta/2 < \delta$, so $|F(y)| < 1$. This means that

$$|F(x)| = \left|F\left(\frac{2\|x\|}{\delta} y\right)\right| = \frac{2\|x\|}{\delta}|F(y)| < \frac{2\|x\|}{\delta},$$

so (iii) holds with $M = 2/\delta$.

$\boxed{(iii) \Rightarrow (iv)}$: Obvious

$\boxed{(iv) \Rightarrow (iii)}$: Denote the supremum in (iv) by M. Then, for all $x \in E$,

$$|F(x)| = F\left(\|x\| \frac{x}{\|x\|}\right) \underset{\text{linearity}}{=} \|x\| \, F\left(\frac{x}{\|x\|}\right) \underset{\text{by (iv)}}{\le} \|x\| \, M.$$

$\boxed{(iii) \Rightarrow (i)}$: Let $\varepsilon > 0$ be arbitrary. Define $\delta = \frac{\varepsilon}{2M}$. Then for all $x, y \in E$,

$$|F(x) - F(y)| \underset{\text{linearity}}{=} |F(x-y)| \underset{\text{by (iii)}}{\le} M\|x-y\|$$

If we now take $\|x-y\| < \delta$, then

$$|F(x) - F(y)| \le M\delta = \frac{\varepsilon}{2\delta}\delta = \varepsilon/2 < \varepsilon,$$

and this shows that (i) holds.

VI. 4. The Dual of a normed space

Warning: Not all functionals are continuous with respect to all possible norms.

6.5 The Fundamental Duality Thm. Let E be a normed space with norm $\|\cdot\|_E$. Let E^* consist of all continuous linear functionals $E \to K$ (where $K = \mathbb{R}$ or $K = \mathbb{C}$). This is a vector space, if we define (for all $F \in E^*$, $G \in E^*$ and all $\lambda \in K$)

$$\begin{cases} (F+G)(x) = F(x) + G(x), & x \in E \\ (\lambda F)(x) = \lambda F(x), & x \in E. \end{cases}$$

Moreover, if we define

$$\|F\|_{E^*} = \sup_{\substack{x \in E \\ \|x\| \le 1}} |F(x)|,$$

then $\|\cdot\|_{E^*}$ is a norm on this space. Finally, E^* (with this norm) is complete, i.e. it is a Banach space.

(This was also proved in Analysis II).

Proof. (i) "Vector space" is easy (a mechanical computation).
(ii) Norm? We must check 3 things
(N1) $\|F\|_{E^*} > 0$ if $F \neq 0$
(N2) $\|\lambda F\|_{E^*} = |\lambda| \, \|F\|_{E^*}$
(N3) $\|F+G\|_{E^*} \le \|F\|_{E^*} + \|G\|_{E^*}$.

First (N1): Obviously $\|F\|_{E^*} \ge 0$. Suppose $= 0$. Then $|F(x)| = 0$ for all $x \in E$ with $\|x\| \le 1$. If $\|x\| > 1$, then we put $e = x/\|x\|$ and set

$$F(x) = F(\|x\| e) = \|x\| F(e) = 0,$$

so $F(x) = 0$ for all $x \in E$. This means that F is the zero function.