II.4 Real vector spaces

To this point we have mainly been discussing *complex* vector spaces.

Defn. A *real* vector space, a *real* inner product space (also called a *Euclidean* space), and a *real* normed space is defined in the same way as the corresponding complex spaces, but we only multiply the vectors by *real numbers*. The set of numbers that we are multiplying the vectors with are called *scalars*. Thus, a

- real vector space = a space with *real* scalars
- (complex) vector space = a space with *complex* scalars.

In the real case *inner* products take *real* values, and the condition

$$(x, y) = \overline{(y, x)} \quad \text{becomes} \quad \boxed{(x, y) = (y, x)}.$$

Note: Most results are true for both real and complex spaces, but sometimes we *must* use one or the other. For example, the eigenvalues of a real matrix may be complex!

When we compute *eigenvalues* and *eigenvectors* we throughout replace $\mathbb{R}^n$ by $\mathbb{C}^n$.

III Hilbert and Banach Spaces

III.1 Cauchy sequences need not converge

In $\mathbb{R}^n$ and $\mathbb{C}^n$ it is true that

a sequence $\{x^k\}_{k=1}^{\infty}$ converges $\Leftrightarrow$ this sequence is a Cauchy sequence.

3.1a Defn. A sequence $\{x^k\}_{k=1}^{\infty}$ in a normed space E is a Cauchy sequence if for every $\varepsilon > 0$ there exists a k_0 such that $\|x^k - x^l\| < \varepsilon$ for $k, l \geq k_0$.

Alternative notation: $\lim\limits_{\substack{k\to\infty \\ l\to\infty}} \|x^k - x^l\| = 0.$

Note: There is no "limit value" present in this definition, to which x^k converges.

Claim: There do exist inner product spaces with the property that *not every* Cauchy sequence has a limit in this space.

Ex. Recall the space RH^2 defined on page 5: Rational functions whose pole lie *outside* of the closed unit disk $\overline{D}$. In this space we can construct a Cauchy sequence which does not converge as follows:

We know from complex function theory (or Analysis II) that the analytic function $f(z) = e^z$ can be approximated by a McLauren series

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k.$$

and the convergence is uniform in every bounded subset of $\mathbb{C}$.

Let CH^2 = the set of all functions which are analytic in some neighborhood of $\overline{D}$, and define the same inner product in this space as in RH^2:

$$(f,g) = \frac{1}{2\pi i}\oint_{\partial D} f(z)\,\overline{g(z)}\,\frac{dz}{z}$$

(cf. page 5). Define $p_n(z) = \sum_{k=0}^{n} \frac{1}{k!} z^k$.

Then each p_n is a polynomial, so it belongs to RH^2. By the "same" computation as on p. 11, since $p_n(z) \to e^z$ uniformly on the unit circle we have

$$\|p_n - f\|_{CH^2} = \frac{1}{2\pi}\int_{\varphi=0}^{2\pi} |p_n(e^{i\varphi}) - e^{i\varphi}|\,d\varphi$$

$\to 0$ as $n \to \infty$.

By the triangle inequality,

$$\|p_n - p_\ell\|_{CH^2} \le \|p_n - f\|_{CH^2} + \|f - p_\ell\|_{CH^2}$$

$\to 0$ as both $n, \ell \to \infty$.

The norm in CH^2 is the same as the norm in RH^2, so we find that $\{p_n\}_{n=0}^{\infty}$ is a Cauchy sequence in RH^2. But it does not converge in RH^2: The only possible limit is e^z, and this function is not rational, so

$$\boxed{\text{the Cauchy sequence } p_n(z) = \sum_{k=0}^{n} \frac{1}{k!} z^k \text{ does not have a limit in } RH^2}$$

(Problem 3.2)

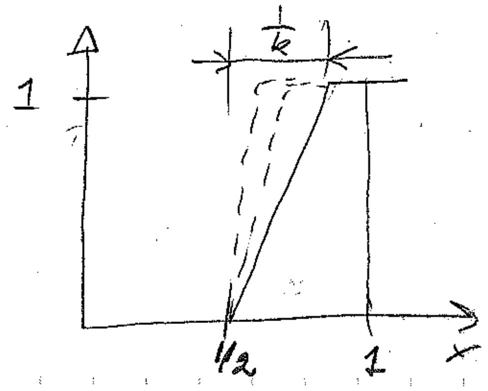
Ex. The space $C[0,1]$ is an inner product space with the inner product

$$(f,g) = \int_0^1 f(t)\,\overline{g(t)}\,dt.$$

This space also contains Cauchy sequences which do not converge.

In this space we can take the following sequence of functions (with $k \ge 3$)

$$f^k(x) = \begin{cases} 0, & 0 \le x < 1/2, \\ k(x - 1/2), & 1/2 \le x < \frac{1}{2} + \frac{1}{k}, \\ 1, & \frac{1}{2} + \frac{1}{k} \le x \le 1. \end{cases}$$



Clearly, at each point $x \in [0,1]$, this sequence converges to the function

$$f(x) = \begin{cases} 0, & 0 \le x \le 1/2. \\ 1, & x > 1/2. \end{cases}$$

This function is not in $C[0,1]$ since it is not continuous.

Claim 1: The restriction of f^k to the interval $[0, \frac{1}{2}]$ converges to the zero function in $C[0,\frac{1}{2}]$. Proof obvious.

Claim 2: The restriction of f^k to $[\frac{1}{2}, 1]$ converges to the function $g(x) \equiv 1$ in the space $C[\frac{1}{2}, 0]$.

Proof:

$$\|f^k - g\|^2_{C[\frac{1}{2},1]} = \int_{1/2}^{1} |f^k(x) - g(x)|^2\,dx$$

$$= \int_{1/2}^{1/2+1/k} |k(x-\tfrac{1}{2}) - 1|^2\,dx \qquad \left(\begin{array}{l} k(x-\frac{1}{2}) = y \\ k\,dx = dy \end{array}\right)$$

$$= \int_0^1 |y-1|^2\,\frac{dy}{k} = \frac{1}{k}\int_0^1 (y-1)^2\,dy \to 0 \text{ as } k \to \infty.$$

Claim 3: ϕ^k does not have a limit in $C[0,1]$.

Proof: If ϕ^k converges to a continuous function $g \in C[0,1]$, then it follows from Claims 1 and 2 that $g(x) = 0$ for $0 \le x \le 1/2$ and that $g(x) = 1$ for $1/2 \le x \le 1$. This is impossible!

Claim 4: ϕ^k is a Cauchy sequence.

Proof: By Claims 1 and 2 (since every convergent sequence is Cauchy),

$$\lim_{\substack{k\to\infty \\ l\to\infty}} \int_0^{1/2} |\phi^k(x) - \phi^l(x)|^2 \, dx = 0 \quad \text{and}$$

$$\lim_{\substack{k\to\infty \\ l\to\infty}} \int_{1/2}^{1} |\phi^k(x) - \phi^l(x)|^2 \, dx = 0 .$$

Thus $\lim_{\substack{k\to\infty \\ l\to\infty}} \int_0^1 |\phi^k(x) - \phi^l(x)|^2 dx = 0$, so this is a Cauchy sequence. $\square$

III.2 Completeness

The property "every Cauchy sequence converges" is so important that we give it a own name.

3.4 Defn. a) A normed space E is <u>complete</u> if every Cauchy sequence converges. Such a space is called a <u>Banach</u> space.

b) A <u>Hilbert space</u> is a <u>complete</u> inner <u>product space</u>, that is, an inner product space with the property that when we interpret it as a normed space with the norm $\|x\| = \sqrt{\langle x, x\rangle}$, then this normed space is complete (= a Banach space).

<u>Recall:</u> $\boxed{l^2}$ = set of complex-valued sequences $\{x_k\}_{k=1}^{\infty}$ satisfying $\sum_{k=1}^{\infty} |x_k|^2 < \infty$ is an inner product space with

$$(x,y)_{l^2} = \sum_{k=1}^{\infty} x_k \overline{y_k}$$

$\boxed{l^\infty}$ = set of complex-valued sequences $\{x_k\}_{k=1}^{\infty}$ satisfying $\sup_{k\ge 0} |x_k| < \infty$ is a normed space with norm

$$\|x\|_{l^\infty} = \sup_{k\ge 1} |x_k|$$

$\boxed{c_0}$ = set of complex-valued sequences $\{x_k\}_{k=1}^{\infty}$ satisfying $\lim_{k\to\infty} x_k = 0$ is a normed space with

$$\|x\|_{c_0} = \max_{k\ge 1} |x_k| .$$

3.2 Thm. a) $\mathbb{C}^n$ and l^2 are Hilbert spaces

b) l^∞ and c_0 are Banach spaces.

Proofs: $\mathbb{C}^n$: See course in multidimensional analysis.
l^2: This was proved in Analysis II. A proof is also given in the book.
l^∞: Proved in Analysis II (fairly straightforward)
c_0: Same as l^∞. $\square$

III.3 The space $L^2(a,b)$

We know that the space $C[a,b]$ is an inner product space with the inner product and norm

$$(\phi,g)=\int_a^b \phi(x)\overline{g(x)}\,dx, \quad \|\phi\|=\left(\int_a^b|\phi(x)|^2dx\right)^{1/2}.$$

However, it is not complete, hence not a Hilbert space. To get a Hilbert space we must "add all those functions which are "limits" of sequences in $C[a,b]$ but do not belong to $C[a,b]$, for example, all "step functions". The exact construction of this set of functions is too complicated to be presented here. It consists of all "Lebesgue measurable functions" on the interval (a,b) which satisfy

$$(*)\quad \int_a^b|\phi(x)|^2\,dx<\infty.$$

In this space we can still use the same candidate for inner product, namely

$$(**)\quad (\phi,g)=\int_a^b\phi(x)\overline{g(x)}\,dx$$

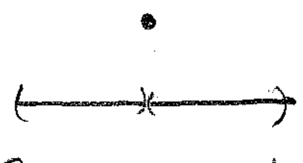
(the integral is interpreted in the Lebesgue sense).

Lemma (**) is a semi-inner product in the space of all Lebesgue measurable functions which satisfy (*). In other words, it has all the other properties of an inner product, but

$$\|\phi\|^2=(\phi,\phi)=0 \not\Rightarrow \phi\equiv 0.$$

(there are nonzero functions with a zero seminorm). For example

$$\phi(x)=\begin{cases}1, & x=\frac{a+b}{2}\\ 0, & \text{otherwise}\end{cases}$$



To overcome this difficulty we use the following "trick": we consider two functions ϕ and g to be "equivalent" whenever $\|\phi-g\|=0$. There is another description of this: ϕ and g are "equivalent" iff $\phi(x)=g(x)$ in "almost all points", i.e., the set $E=\{x\in(a,b)\mid \phi(x)\neq g(x)\}$ has "measure zero". (Think: it contains approximately only countably many points).

3.5 Defn. The space $L^2(a,b)$ consists of all Lebesgue measurable functions on (a,b) satisfying $\int_a^b|\phi(x)|^2dx<\infty$, and we identify any two functions ϕ and g which satisfy $\int_a^b|\phi(x)-g(x)|^2dx=0$.

Throughout the rest of this course

We assume that all the functions ϕ that we encounter are Lebesgue measurable

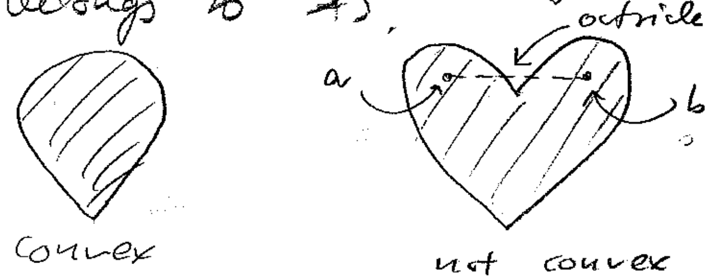
(non-measurable functions are difficult to construct)

III.4 Convex Sets and the closest point

3.7 Defn. A subset A of a real or complex vector space is convex if for all $a, b \in A$ we have

$$\boxed{\lambda a + (1-\lambda) b \in A, \quad a \le \lambda \le b.}$$

(the line joining any two points in A belongs to A).



("Nice-looking" sets are not convex)

3.8 Thm (the most important theorem in optimization theory ??) Let A be a nonempty closed and convex set in a Hilbert space H. Then, for each $x \in H$ there is a unique point $y \in A$ which is closest to x.

In other the function $f(a) = \|x-a\|$ has a unique (global) minimum in the set A, and this minimum is attained at exactly one point $y \in A$. Thus, for all other $a \in A$,

$$\|x-a\| > \|x-y\|.$$

We write this as

$$\|x-y\| = \inf_{a \in A} \|x-a\| = \min_{a\in A} \|x-a\|.$$

Note: This theorem is not true if we
- replace Hilbert by Banach
- replace Hilbert by "inner product space"
- remove "closed"
- remove "convex"
- remove "nonempty"

In this sense it is "best possible".

(Ψ) **Proof.** Let $\boxed{M = \inf_{a \in A} \|x-a\|.}$ Then $0 \le M < \infty$, and for every $n > 0$ there is a y_n such that

(⊛) $\|x-y_n\|^2 < M^2 + \frac{1}{n}$.

Claim: $\{y_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

Proof of claim: By the parallellogram law, with $z_n = x - y_n$,

$$\|z_n - z_m\|^2 + \|z_n + z_m\|^2 = 2\|z_n\|^2 + 2\|z_m\|^2 \iff$$

$$\|y_n - y_m\|^2 + \|2x - y_n - y_m\|^2 = 2\|x-y_n\|^2 + 2\|x-y_m\|^2.$$

$$\le 2\left(M^2 + \frac{1}{n} + M^2 + \frac{1}{m}\right) \iff$$

$$\|y_n - y_m\|^2 \le 4M^2 + 2\left(\frac{1}{n} + \frac{1}{m}\right) - 4\left\|x - \frac{y_n+y_m}{2}\right\|^2.$$

Since A is convex and $\frac{y_n+y_m}{2}$ is the mid-point between y_n and y_m, we have $\frac{y_n+y_m}{2} \in A$, so

$$\left\|x - \frac{y_n+y_m}{2}\right\|^2 \ge M^2, \text{ and we get}$$

$$\|y_n - y_m\|^2 \le 2\left(\frac{1}{n} + \frac{1}{m}\right) \to 0 \text{ as } n, m \to \infty.$$

Thus, $\{y_n\}_{n=1}^{\infty}$ is a Cauchy-sequence.
H being complete, $y_n \to y$ for some $y \in H$.
Since A is closed, we must have $y \in A$.
The norm-function is continuous (see Ex. 14, p. 14), hence $\|x-y\| = \lim_{n\to\infty} \|x-y_n\|$ (in particular, this limit exists). Because of ⊛ we must have $\|x-y\| \le M$. But since $y \in A$ we must also have $\|x-y\| \ge M = \inf_{a\in A} \|x-a\|$. Thus, $\boxed{\|x-y\| = M}$, and the function $a \mapsto \|x-a\|$ achieves its minimum at $a = y$.

by (Ψ) above

Uniqueness: Assume that both $\|x-y\| = M$ and $\|x-z\| = M$. By the parallellogram law applied to $v = x-y$ and $w = x-z$ we get

$$\|v-w\|^2 + \|v+w\|^2 = 2\|w\|^2 + 2\|v\|^2 \iff$$

$$\|y-z\|^2 = 4M^2 - \|2x-(y+z)\|^2 = 4\left(M^2 - \left\|x - \frac{y+z}{2}\right\|^2\right)$$

≤ 0 since $\frac{y+z}{2} \in A \Rightarrow y = z$. □

IV Orthogonal expansions

IV.1 Orthogonality and Fourier Coefficients

4.1a Defn. i) Two vectors x, y in an inner product space are orthogonal iff $(x,y)=0$. We then write $x \perp y$.

ii) If $x \neq 0$ and $y \neq 0$, then the angle φ between x and y is

$$\varphi = \arccos \frac{(x,y)}{\|x\|\|y\|} \qquad (\text{and } 0 \le \varphi \le \pi).$$

Note that $x \perp y \Leftrightarrow \varphi = \pi/2$ (provided $x \neq 0$ and $y \neq 0$). Also: $x \uparrow\uparrow y \Leftrightarrow \varphi = 0$ and $x \uparrow\downarrow y \Leftrightarrow \varphi = \pi$.

4.1b. A collection of vectors $\{e_\alpha\}_{\alpha \in A}$ in a inner product space V is an orthogonal system if

i) every $e_\alpha \neq 0$

ii) $e_\alpha \perp e_\beta$ if $\alpha \neq \beta$.

This system is orthonormal if, in addition

iii) $\|e_\alpha\| = 1$ for all $\alpha \in A$.

If the index set A is $\mathbb{N} = \{1, 2, 3 \dots\}$, then we call this an orthogonal or orthonormal sequence.

(sometimes we instead use the index set $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ as an index set, and still call this a (bilateral) sequence.

4.2a Ex. Define $e^k = \{\delta_n^k\}_{n=1}^\infty$, where $\delta_n^k = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$. This is an orthonormal sequence in ℓ^2.

$e_1 = \{1, 0, 0, 0, 0 \dots\}$ $\quad e_2 = \{0, 1, 0, \dots\}$

$e_3 = \{0, 0, 1, 0, \dots\}$ etc.

4.2b Ex. Put $e_n(t) = e^{2\pi i n t}$, $n = 0, \pm 1, \pm 2, \dots$. This is an orthonormal sequence in $L^2(0,1)$.

4.2c Ex. z^n, $n = 0, 1, 2, \dots$ is an orthonorm. seq. in RH^2

4.2d Ex. The collection of functions

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos t, \frac{1}{\sqrt{\pi}} \sin t, \frac{1}{\sqrt{\pi}} \cos(2t), \frac{1}{\sqrt{\pi}} \sin(2t), \dots \right\}$$

$$= \left\{ \frac{1}{\sqrt{2\pi}} \right\} \cup \left\{ \frac{1}{\sqrt{\pi}} \cos nt \right\}_{n=1}^\infty \cup \left\{ \frac{1}{\sqrt{\pi}} \sin nt \right\}_{n=1}^\infty$$

is an orthonormal sequence in $L^2(-\pi, \pi)$, and also in $L^2(0, 2\pi)$.

4.2e Ex z^n, $n = 0, \pm 1, \pm 2, \dots$ is an orthonorm. seq. in RL^2

We borrow the following terminology from Fourier analysis:

4.3 Defn. If $\{e_n\}_{n=1}^\infty$ is an orthonormal sequence in a inner product space V and $x \in V$, then we call (x, e_n) the n:th Fourier coefficient of x. The (formal) Fourier series of x with respect to $\{e_n\}_{n=1}^\infty$ is the (formal) sum

$$\sum_{n=1}^\infty (x, e_n) e_n .$$

$\boxed{\text{Note}}$ At this stage we do not yet claim that this sum converges! However, the sequence of Fourier coefficients $\{(x, e_n)\}_{n=1}^\infty$ is always well defined.

Note: In the complex case you must not replace (x, e_n) by (e_n, x)!! (In the real case they are equal.)

4.4 Thm (Pythagoras theorem). If $x_1, x_2, \dots, x_n$ is an orthogonal system, then

$$\left\| \sum_{j=1}^n x_j \right\|^2 = \sum_{j=1}^n \|x_j\|^2 .$$

(square of diagonal = sum of squares of "katelerna".

Proof. $\|\sum_{j=1}^{n} x_j\|^2 = \left(\sum_{i=1}^{n} x_i, \sum_{j=1}^{n} x_j\right)$
$= \sum_{i=1}^{n}\sum_{j=1}^{n} (x_i, x_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} \delta_i^j \|x_i\|^2 = \sum_{i=1}^{n} \|x_i\|^2$. □

IV. 2 Least square minimization

Obviously, every subspace F of a vector space E is convex, and if it is also finite-dimensional, then it is closed. We can apply Theorem 3.8 to get a least square minimization result. We begin by choosing an orthonormal basis $\{e_j\}_{j=1}^{n}$ in the subspace F. Then every $y \in F$ can be written in the form $y = \sum_{j=1}^{n} \lambda_j e_j$ for some scalars (or "coordinates") $\lambda_j \in \mathbb{C}$ or $\lambda_j \in \mathbb{R}$.

4.5. Lemma Let $e_1, \dots, e_n$ be an orthonormal system in an inner product space H, and let $\lambda_j \in \mathbb{C}$, $1 \le j \le n$, $x \in H$. Then

$$(*) \quad \|x - \sum_{j=1}^{n} \lambda_j e_j\|^2 = \|x\|^2 + \sum_{j=1}^{n} |\lambda_j - c_j|^2 - \sum_{j=1}^{n} |c_j|^2$$

where $c_j = (x, e_j)$ is the j-th Fourier coefficient.

(Note: This is related to the "Steiner rule" in statistics).

Proof: Easy computation. See the book. (Suggestion: Replace one of the indexes j by i. This makes it simpler!) so that

$$\|x - \sum \lambda_j e_j\|^2 = \left(x - \sum_i \lambda_i e_i, x - \sum_j \lambda_j e_j\right) \text{ etc...}$$

Clearly, we get the minimum of the function in $(*)$ with respect to $\lambda_1, \lambda_2, \dots, \lambda_n$ by choosing $\lambda_j = c_j$, $1 \le j \le n$. This gives: (next page)

4.6 Thm. Let $e_1, \dots, e_n$ be an orthonormal system in an inner product space E, and $x \in E$. Denote the linear span of $\{e_1, e_2, \dots, e_n\}$ by F. Then the closest point y to x in F is given by

$$y = \sum_{j=1}^{n} (x, e_j) e_j,$$

and the distance from x to F is given by

$$d^2 = \|x\|^2 - \sum_{j=1}^{n} |(x, e_j)|^2.$$

(Note: (x, e_j) are the Fourier coefficients.)

4.7 Corollary. In the special case where $x \in F$, we get

$$x = \sum_{j=1}^{n} (x, e_j) e_j$$

(because $y = x$ and $d = 0$ in this case).

4.8 Bessel's inequality Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal sequence in an inner product space H, and let $x \in H$. Then

$$(*) \quad \sum_{n=1}^{\infty} |(x, e_n)|^2 \le \|x\|^2$$

(here $c_n = (x, e_n)$ are the Fourier coeff. of x). In particular, $\{c_n\}_{n=1}^{\infty} \in \ell^2$.

Proof easy. See book.

Note: In this chapter we have not yet used the completeness of the space (the "Hilbert" property). This is needed when we want to let $n \to \infty$. We would like to get equality in $(*)$, because that would give us a "infinite-dimensional Pythagoras' theorem". For that we need completeness!

IV 3 Convergence of orthogonal sums

4.10 Defn. The (formal) series $\sum_{n=1}^{\infty} x_n$ converges and its sum is S iff

$$\lim_{k\to\infty} \sum_{n=1}^{k} x_n = S.$$

(i.e., this limit exists and is equal to S).
In this case we write

$$S = \sum_{n=1}^{\infty} x_n.$$

"obetingad", "ovillkorlig"

This sum converges absolutely if $\sum_{n=1}^{\infty} \|x_n\| < \infty$.
It converges unconditionally if we can rearrange the terms in an arbitrary way, and the rearranged series still converges to the same limit.
Thus, if $j \to n_j$ is an arbitrary mapping of $\mathbb{N}$ onto $\mathbb{N}$, then we still have

$$\lim_{k\to\infty} \sum_{j=1}^{k} x_{n_j} = S.$$

Note: According to Analysis II,
convergence $\not\Rightarrow$ absolute convergence, but
absolute convergence $\Rightarrow$ convergence in a complete space

According to analysis & multidim-analysis:
in $\mathbb{R}^n$ and $\mathbb{C}^n$; absolute conv $\Leftrightarrow$ uncondit. conv.

Right now we are interested in orthogonal sums where $x_n \perp x_m$ for $n \neq m$. (note: requires dimension to be ∞; trivial otherwise).

4.11 Thm Let $\{e_n\}$ be an orthonormal sequence in a Hilbert space H, and let $\lambda_n \in \mathbb{C}$. Then t.f.c.a.e. (... are equivalent) (next page)

(i) $\sum_{n=1}^{\infty} |\lambda_n|^2 < \infty$

(ii) $\sum_{n=1}^{\infty} \lambda_n e_n$ converges

(iii) $\sum_{n=1}^{\infty} \lambda_n e_n$ converges unconditionally.

Proof: (i) $\Leftrightarrow$ (ii): See the book, pp. 34–35.
(iii) $\Rightarrow$ (ii): Trivial
(ii) $\Rightarrow$ (iii): Since (ii) $\Leftrightarrow$ (i), and since the sum in (i) converges absolutely, it still converges if we rearrange the terms, so every rearranged sum $\sum_{j=1}^{\infty} \lambda_{n_j} e_{n_j}$. Is the sum the same? Yes:
Idea: Put $\sum_{n=1}^{\infty} \lambda_n e_n = x$. Take N so large that $\|x_N - x\| < \varepsilon$ where $x_N = \sum_{n=1}^{N} \lambda_n e_n$.
By orthogonality,

$$\varepsilon \geq \|x_N - x\|^2 = \Big\|\sum_{n=N+1}^{\infty} \lambda_n e_n\Big\|^2 = \sum_{n=N+1}^{\infty} |\lambda_n|^2.$$

Now rearrange the series into $\sum_{j=1}^{\infty} \lambda_{n_j} e_{n_j}$. Take K so large that every index n with $n \le N$ is contained in the set $\{n_j \mid 1 \le j \le K\}$. Denote $y = \sum_{j=1}^{\infty} \lambda_{n_j} e_{n_j}$, and $y_K = \sum_{j=1}^{K} \lambda_{n_j} e_{n_j}$.
Then (prove this!)

$\|y - y_K\| \le \varepsilon$, $\|y_K - x_N\| \le \varepsilon$, hence

$$\|y - x\| \le \|y - y_K\| + \|y_K - x_N\| + \|x_N - x\| \le 3\varepsilon. \quad \square$$

This theorem can be reformulated (put $\lambda_n e_n = x_n$).

4.11a Cor: Let $\{x_n\}_{n=1}^{\infty}$ be an orthogonal sequence in a Hilbert space. Then

$$\sum_{n=1}^{\infty} x_n \text{ converges} \Leftrightarrow \sum_{n=1}^{\infty} \|x_n\|^2 < \infty,$$

and the convergence is unconditional.

Proof: Apply Thm 4.11 with $e_n = x_n/\|x_n\|$, $\lambda_n = \|x_n\|$. $\square$

Warning: Not true without orthogonality.
(take, e.g., $E = \mathbb{R}$ or $E = \mathbb{C}$).

IV. 4 Complete orthonormal sets, a basis

Let us recapitulate: If $x \in H$, and if $\{e_n\}_{n=1}^{\infty}$ is orthonormal, then by Bessel's inequality, the Fourier coefficients satisfy:

$$\sum_{n=1}^{\infty} |(x, e_n)|^2 < \infty.$$

The closest point to x in the subspace spanned by $\{e_1, e_2, \ldots, e_n\}$ is

$$x_n = \sum_{j=1}^{n} (x, e_j) e_j \quad .$$

By Thm. 4.11, if H is a Hilbert space, the

$$\boxed{\text{the sum } \sum_{j=1}^{\infty} (x, e_j) e_j \text{ converges}} \; ;$$

i.e., $\lim_{n\to\infty} x_n$ exists. However, it need not converge to x. For example, we could have $x \perp e_n$ for all n, in which case $x_n = 0$ for all n!

4.13 Defn. An orthonormal sequence $\{e_n\}_1^{\infty}$ in a Hilbert space H is complete (or maximal) iff the only vector which is orthogonal to all the vectors e_n is the zero vector. In this case we call $\{e_n\}$ an orthonormal basis.

Thus, $\{e_n\}$ is complete iff the following is true:

$$\boxed{x = 0 \iff (x, e_n) = 0 \text{ for all } n}$$

Note: This is a different use of the world "complete" which does not refer to Cauchy sequences.

4.14 Thm If $\{e_n\}$ is a complete orthonormal sequence in a Hilbert space and if $x \in H$, then

$$(*) \qquad x = \sum_{n=1}^{\infty} (x, e_n) e_n \qquad \text{and}$$

$$(**) \qquad \|x\|^2 = \sum_{n=1}^{\infty} |(x, e_n)|^2$$

Note: (**) is an infinite-dimensional Pythagoras theorem.

Note: Still true if we replace "Hilbert" by "inner product", but the proof is more difficult.

Proof: We observed above that $\lim_{n\to\infty} x_n$ exists, where $x_n = \sum_{j=1}^{n} (x, e_j) e_j$. Call the limit y. By the continuity of the inner product, for all k,

$$(y - x, e_k) = (\lim_{n\to\infty} x_n, e_k) - (x, e_k)$$

$$= \lim_{n\to\infty} \underbrace{\left(\sum_{j=1}^{n} (x, e_j) e_j, e_k \right)}_{= \begin{cases} 0, & \text{if } n < k \\ (x, e_k) & \text{if } n \geq k \end{cases}} - (x, e_k)$$

(by continuity of inner prod)

$$= 0.$$

Thus $y - x \perp e_n$ for all n, so by the completeness of $\{e_n\}$, $y - x = 0$, i.e., $y = x$. This proves (*).

(**) Follows from the continuity of the norm: since $x_n \to x$, we must have $\|x_n\|^2 \to \|x\|^2$, and

$$\|x_n\|^2 = \sum_{j=1}^{n} |(x, e_j)|^2 \to \sum_{j=1}^{\infty} |(x, e_j)|^2 \quad n \to \infty.$$

∎