$$\frac{\phi''(x)}{\phi(x)} = \frac{T'(t)}{kT(t)},$$

Here left hand side does not depend on t,
right — — — — on x,

so $\frac{\phi''(x)}{\phi(x)} = \frac{T'(t)}{kT(t)}$ depends on neither x nor t, i.e., it is a constant: $(=-\lambda)$

$$\frac{\phi''(x)}{\phi(x)} = -\lambda = \frac{T'(t)}{kT(t)}, \quad \Rightarrow$$

(11.4) $\phi''(x) + \lambda\phi(x) = 0$

(11.5) $T'(t) + k\lambda T(t) = 0$

First solve (11.4): This combined with (11.2) gives a Sturm-Liouville system:

$$\phi'' + \lambda f = 0$$
$$\phi'(0) = 0 = \phi'(L).$$

A "straightforward computation" gives: the eigenvalues are

Note: zero is an eigenvalue

$$\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, 3 \ldots,$$

$$\boxed{\varphi_n(x) = A_n \cos\left(\frac{n\pi x}{L}\right).}$$

Normalization: $\int_0^L \rho(x)|\varphi_n(x)|^2 dx = 1$, or

$$A_n^2 \int_0^L \cos^2\left(\frac{n\pi x}{L}\right) dx = 1 \Rightarrow A_n = \sqrt{2/L} \quad (\text{if } n \geq 1)$$

and

$$\boxed{\begin{aligned} &\lambda_n = \frac{n^2\pi^2}{L^2}, \quad n = 0, 1, 2, 3, 4, \ldots \\ &\varphi_n = \sqrt{2/L}\cos\left(\frac{n\pi x}{L}\right), \quad n \geq 1 \\ &\varphi_0 \equiv \frac{1}{\sqrt{L}} \end{aligned}}$$

Note that if we divide ϕ by a constant k and divide T by the same constant, then $u(x,t) = \phi(x)T(t)$ does not change. Thus we may "without loss of generality" normalize φ_n as above.

Next we look at the corresponding equation (11.5) with $\lambda = \lambda_n$ (put $T \to T_n$)

$$T_n'(t) + k\lambda_n T_n(t) = 0 \Rightarrow$$
$$T_n(t) = T_n(0)e^{-k\lambda_n t}.$$

Thus, the corresponding function

$$u_n(x,t) = \varphi_n(x)T_n(t)$$
$$= \underbrace{A_n T_n(0)}_{\text{call this } B_n} e^{-n^2\pi^2 kt/L} \cos\left(\frac{n\pi x}{L}\right)$$

$$\boxed{u_n(x,t) = B_n e^{-n^2\pi^2 kt/L}\cos\left(\frac{n\pi x}{L}\right)}$$

Question: Can we choose the coefficients B_n so that the solution of (11.1) – (11.3) is given by

$$(*) \quad u(x,t) = \sum_{n=0}^{\infty} B_n e^{-n^2\pi^2 kt/L} \cos\left(\frac{n\pi x}{L}\right)$$

Obviously the constants B_n must depend on the initial function h in (11.3); we have already taken care of (11.1) and (11.2).

If the answer is "yes", then in which sense does the sum converge?

Answer: By Thm. 11.1, $\{\varphi_n\}_{n=0}^{\infty}$ is a complete orthonormal sequence in $L^2(0,L)$. (note that $\rho(x) \equiv 1$). Take $t=0$ in $(*)$, and use (11.3):

$$h(x) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi x}{L}\right),$$

where $B_n = A_n T_n(0)$, so.

$$w(x) = \sum_{n=1}^{\infty} T_n(0)\, \varphi_n(x).$$

This is a Fourier series (the sequence $\{\varphi_n\}$ is orthonormal and complete), so

$$T_n(0) = (w, \varphi_n) = \int_0^L h(x)\, A_n \cos\left(\frac{n\pi x}{L}\right) dx,$$

and
$$B_n = A_n T_n(0) = \begin{cases} \frac{1}{L}\int_0^L h(x)\,dx, & n=0, \\ \frac{2}{L}\int_0^L h(x)\cos\left(\frac{n\pi x}{L}\right)dx, & n \geq 1. \end{cases}$$

Thus, if everything that we have done can be justified, then

$$(*) \quad \boxed{\begin{aligned} u(x,t) &= \frac{1}{L}\int_0^L h(x)\,dx + \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 kt/L} \cos\left(\frac{n\pi x}{L}\right)dx, \\ \text{where } B_n &= \frac{2}{L}\int_0^L h(x)\cos\left(\frac{n\pi x}{L}\right)dx \end{aligned}}$$

Indeed, with some work, it is possible to prove that this is correct:

Thm 11.3. Suppose that h is two times continuously differentiable. Then the system of equations (11.1) – (11.3) has at most one solution u which is two times cont. differentiable, and it must be given (if it exists) by $(*)$. The convergence in $(*)$ is of the following type: For every fixed $t \geq 0$ the right hand side converges in $L^2(0,L)$ to $u(x,t)$.

Note: Here we had $\rho(x) \equiv 1$. Otherwise we would have got convergence of $\sqrt{\rho}\, u_n(x,t)$ to $\sqrt{\rho}\, u(x,t)$ in $L^2(0,L)$. This is not important in the regular case, but it is important in the singular case. In the regular case,

$\sqrt{\rho}\, u_n(x,t) \to \sqrt{\rho}\, u(x,t)$ (in $L^2(0,L)$; t fixed) iff $u_n(x,t) \to u(x,t)$ since both $\sqrt{\rho(x)}$ and $1/\sqrt{\rho(x)}$ are bounded.

Proof (outline): Suppose that we have a solution $u(x,t)$. Put (recall that $\rho(x) \equiv 1$)

$$c_n(t) = (u(\cdot,t), \varphi_n) = \int_0^L u(x,t)\, \overline{\varphi_n(x)}\, dx. \quad \left(= T_n(t)\right)$$

Since $\{\varphi_n\}_{n=0}^{\infty}$ is a complete orthonormal sequence, for each fixed t,

$$\lim_{N\to\infty} \sum_{n=0}^{N} c_n(t)\varphi_n(x) = u(t,x)$$

(the convergence is in $L^2(0,L)$ with respect to x).

By using "standard properties of the Lebesgue integral and some analysis", it is possible to show that we are allowed to differentiate under the integral sign (see the book):

$$c_n'(t) = \int_0^L \frac{\partial}{\partial t} u(x,t)\, \overline{\varphi_n(x)}\, dx.$$

We assumed that u satisfies (11.1), so

$$\frac{\partial}{\partial t} u(x,t) = k \frac{\partial^2}{\partial x^2} u(x,t), \quad \text{and}$$

$$(**) \quad c_n'(t) = \int_0^L k \frac{\partial^2}{\partial x^2} u(x,t)\, \overline{\varphi_n(x)}\, dx.$$

We define the operator $\mathcal{L}$ as usual (for equation (11.4)):

$$\mathcal{L} f = f'', \quad \text{and}$$
$$f \in \mathrm{Dom}(\mathcal{L}) \Rightarrow f'(0) = 0, \; f'(L) = 0.$$

Then $(**)$ becomes (differentiation w.r.t. x, not t)

$$c_n'(t) = k\int_0^L (\mathcal{L}) u(x,t)\, \overline{\varphi_n(x)}\, dx$$
$$= k\,(\mathcal{L} u(\cdot,t), \varphi_n) \qquad (\mathcal{L} \text{ is self-adjoint})$$

$$= k(u(\cdot,t), \Delta\varphi_n) \quad (\varphi_n \text{ is an eigenfunction})$$
$$= k(u(\cdot,t), -\lambda_n\varphi_n)$$
$$= -\lambda_n k(u(\cdot,t), \varphi_n)$$
$$= -\lambda_n k\, c_n(t).$$

Thus, $c_n'(t) = -\lambda_n k\, c_n(t)$, so necessarily $c_n(t) = c_n(0)e^{-\lambda_n k t}$

$$= c_n(0)e^{-n^2\pi^2 kt/L}$$

Thus, we get formula ⊛ on p. 123 (since $c_n(0) = \int_0^L h(x)\varphi_n(x)\,dx$.). ∎

Note: By working harder it is possible to show that the function u in ⊛ always is a solution (at least if g is two times continuously differentiable). Thus, Theorem 11.3 can be simplified into:

Thm 11.4: If g is two times continuously differentiable, then (11.1) − (11.3) has a unique two times continuously differentiable solution, and it is given by ⊛ on p. 113.

Note: This example can be continued: If we add some heating to the bar, distributed over its entire length, the the equation becomes:

$$(11.6) \quad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + g,$$

and (11.2) − (11.3) stay the same. Here $g(t,x)$ is the amount of heat added at the point x at time t. The solution proceeds in the same way, but (11.4) becomes a nonhomogeneous equation $f'' + \lambda f = g$, which is treated as described in section XI.2.

XII Positive Operators

In this chapter we study an extension problem, which is part of a more general extension problem that will be discussed later.

XII.1 Positive operators

12.1 Defn An operator $A \in \mathcal{L}(H)$, where H is a Hilbert space, is positive (= non-negative) if $(Ax, x) \geq 0$ for all $x \in H$. We denote this by $A \geq 0$. The notation $A \geq B$ means that $A - B \geq 0$, i.e.,

$$(Ax, x) \geq (Bx, x) \quad \forall x \in H.$$

We call A strictly positive if $A \geq \varepsilon I$ for some $\varepsilon > 0$, i.e.,

$$(Ax, x) \geq \varepsilon\|x\|^2 \quad \forall x \in H.$$

12.2 Ex. i) $A = 0$ is positive

ii) $A = I$ is strictly positive ($\varepsilon = 1$)

iii) The diagonal operator D in Homework 25 is positive $\Leftrightarrow$ $\lambda_n \geq 0$ for all n. If $\{\varphi_n\}_{n=1}^\infty$ is complete (as an orthogonal sequence), then D is strictly positive iff $\lambda_n \geq \varepsilon > 0$ for all $n \geq 1$.

iv) The operator $(Mx)(t) = t\,x(t)$, $0<t<1$ is positive on $L^2(0,1)$, but not strictly.

v) $A = \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$ is not positive, because

$$\left(A\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right) = \left(\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right)$$
$$= \left(\begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\right) = -1\cdot 1 + 1(-1) = -2 < 0.$$

vi) The matrix $A = \begin{bmatrix}1&1\\1&1\end{bmatrix}$ is positive, because $A = \begin{bmatrix}1\\1\end{bmatrix}[1,1]$,

and $(Ax, x) = \left(\begin{bmatrix}1\\1\end{bmatrix}[1\ 1]x, x\right)$

(the adjoint of $\begin{bmatrix}1\\1\end{bmatrix}$ is $[1\ 1]$)

$= ([1\ 1]x, [1\ 1]x)$

$= \|[1\ 1]x\|^2 = |x_1+x_2|^2 \geq 0$,

(write $x = \begin{pmatrix}x_1\\x_2\end{pmatrix}$).

vii) More generally for all $T \in \mathcal{L}(E,F)$, we have $T^*T \geq 0$, because

$(T^*Tx, x) = (Tx, Tx) = \|Tx\|^2 \geq 0$.

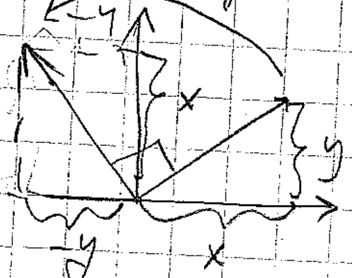
<u>12.3 Lemma</u> Let H be a <u>complex</u> Hilbert space.

i) If $(Ax,x) = 0$ for all $x \in H$, then $A = 0$

ii) If (Ax,x) is <u>real</u> for all $x \in H$, then A is self-adjoint.

In particular, every <u>positive</u> operator on a <u>complex</u> Hilbert space is self-adjoint.

<u>Note</u>: <u>Not</u> <u>true</u> for real Hilbert space. For example $A = \begin{bmatrix}0&-1\\1&0\end{bmatrix}$



(this is a 90° degree rotation)

Then, for each <u>real</u> vector $\begin{pmatrix}x\\y\end{pmatrix}$,

$\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = \left(\begin{pmatrix}-y\\x\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = -yx + xy = 0$.

So we "should have" $A = 0$???

Not true, because for some <u>complex</u> vectors we get a nonzero result:

Take $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\i\end{pmatrix}$.

Then $\overline{\begin{pmatrix}x\\y\end{pmatrix}} = \begin{pmatrix}1\\-i\end{pmatrix}$ and $A\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-i\\1\end{pmatrix}$,

so $\left(A\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}x\\y\end{pmatrix}\right) = (-i)(1) + 1(-i) = -2i \neq 0$.

<u>Proof</u> of Lemma 12.3.

i) For all $\lambda, \mu \in \mathbb{C}$ we have

$$\overbrace{(A(\lambda x+\mu y), \lambda x+\mu y)}^{=0} = |\lambda|^2 \overbrace{(Ax,x)}^{=0}$$

$$+ \lambda\bar{\mu}(Ax,y) + \mu\bar{\lambda}(Ay,x) + |\mu|^2 \overbrace{(Ay,y)}^{=0}.$$

$\lambda = \mu = 1$ gives

(*) $(Ax,y) + (Ay,x) = 0 \quad \forall x,y \in H$.

$\lambda = i, \mu = 1$ gives

$i(Ax,y) - i(Ay,x) = 0 \quad \forall x,y \in H$.

Divide this by i and add to (*) $\Rightarrow$ $(Ax,y) = 0 \quad \forall x \in H, y \in H$.

Take $y = Ax \Rightarrow \|Ax\|^2 = 0 \quad \forall x \in H$ $\Rightarrow A = 0$.

ii) If instead (Ax,x) is real for all x,

$(A^*x, x) = (x, Ax) = \overline{(x, Ax)} = (Ax, x)$ for all $x \in H$, so

$((A^* - A)x, x) = 0 \quad \forall x \in H$.

By i), $A^* - A = 0$, so A is self-adj.

Note. In the real case we can always symmetrize a positive operator:

$$0 \le (Ax,x) = (x, A^*x) = (A^*x, x), \text{ so}$$

$$(Ax,x) + (A^*x,x) = 2(Ax,x), \text{ and}$$

$$(Ax,x) = \left(\tfrac{1}{2}(A+A^*)x, x\right), \text{ where}$$

$\frac{1}{2}(A+A^*)$ is self-adjoint and positive.

Defn: $\frac{1}{2}(A+A^*)$ is the real part of A.

$\frac{1}{2i}(A-A^*)$ is the imaginary part of A.

(In complex we always have $A=A^*$, so $\frac{1}{2}(A+A^*) = A$).

12.4 Thm. Every (self-adjoint) positive operator A on a complex Hilbert space has a unique (self-adjoint) positive square root. We denote this square root by $A^{1/2}$ or $\sqrt{A}$.

More precisely: There is a positive operator B such that $A = B^2 = BB$.

Moreover, it is possible to find a sequence of polynomials p_n so that

$$(*) \quad p_n(A)x \to A^{1/2}x \quad \forall x \in H \text{ as } k \to \infty.$$

Explanation: If $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_mx^m$, then by $p(A)$ we mean

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \dots + a_m A^m.$$

$$= \sum_{k=0}^{m} a_k A^k \quad (\text{recall that } A^0 = I).$$

Proof. The general case is difficult (see the book).

Here: Only a special case: A diagonal operator in a separable Hilbert space (as in Homework 25). Then

(For example compact and self-adjoint)

$$Ax = \sum_{n=1}^{\infty} \lambda_n (x, \varphi_n)\varphi_n \quad \left(\text{compact iff } \lambda_n \to 0,\ n\to\infty\right)$$

for some orthonormal sequence φ_n, and $\lambda_n \ge 0$.

Existence of $A^{1/2}$: Define $Bx = \sum_{n=1}^{\infty} \sqrt{\lambda_n}(x,\varphi_n)\varphi_n$.

Then $B^2 = A$, and B is positive.

Polynomials: Choose any sequence of polynomials p_k so that $p_k(x) \to \sqrt{x}$ uniformly on any finite subinterval of $[0,\infty)$. Put

$$A_k = p_k(A).$$

If $p_k(x) = a_0 + a_1x + a_2x^2 + \dots + a_mx^m$, then

$$A_k = p_k(A) = a_0 I + a_1 A + \dots + a_m x^m, \text{ so}$$

Recall: $Ax = \sum_{n=1}^{\infty} \lambda_n (x,\varphi_n)\varphi_n$, and

$$(Ax, \varphi_n) = \lambda_n (x,\varphi_n)\varphi_n, \text{ so}$$

$$A^2x = \sum_{n=1}^{\infty} \lambda_n (Ax,\varphi_n)\varphi_n$$

$$= \sum_{n=1}^{\infty} \lambda_n^2 (x,\varphi_n)\varphi_n.$$

Continue: $A^3x = A(A^2x) = \sum_{n=1}^{\infty} \lambda_n^3 (x,\varphi_n)\varphi_n$, etc., so

$$p_k(A) = \sum_{n=1}^{\infty} \mu_n (x,\varphi_n)\varphi_n, \text{ where}$$

$\mu_n = p_k(\lambda_n)$. Thus,

Thus, $B - p_k(A)$ is the diagonal op.

$$(B - p_k(A))x = \sum_{n=1}^{\infty} \left(\sqrt{\lambda_n} - p_k(\lambda_n)\right)(x, \varphi_n)\varphi_n,$$

and by homework 25,

$$\|B - p_k(A)\| \le \sup_n |\sqrt{\lambda_n} - p_k(\lambda_n)|.$$

All the (eigenvalues) λ_n belong to the interval $[0, \|A\|]$ and on this interval $p_k(\lambda) - \sqrt{\lambda} \to 0$ (unif. in λ). Thus,

$$\|p_k(A) - B\| \to 0 \quad \text{as } k \to \infty.$$

Uniqueness is slightly more difficult. See the book (pp. 223-224).

XII.2 Contractions

12.5 Defn. An operator $T \in \mathcal{L}(H, K)$ is a contraction if $\|T\| \le 1$, and a strict contraction if $\|T\| < 1$.

Note: Thm 7.10 says: If $A \in \mathcal{L}(E)$ is a strict contraction, then $I - A$ is invertible, and

$$(I-A)^{-1} = \sum_{n=0}^{\infty} A^n.$$

12.6 Thm. The following cond. are equivalent:

(i) T is a contraction
(ii) $T^*T \le I$
(iii) $I - T^*T \ge 0$
(iv) T^* is a contraction
(v) $TT^* \le I$
(vi) $I - TT^* \ge 0$.

Proof: (i) $\Leftrightarrow$ (iv): See Thm 7.15, p. 69.
(ii) $\Leftrightarrow$ (iii): Defn. 12.1
(i) $\Leftrightarrow$ (ii): (i) $\Leftrightarrow \|Tx\|^2 \le \|x\|^2 \ \forall x \in E$
$\Leftrightarrow (Tx, Tx) \le (x, x) \ \forall x \in E$
$\Leftrightarrow (T^*Tx, x) \le (x, x) \ \forall x \in E$
$\Leftrightarrow T^*T \le I = $ (ii). $\square$

XII.3 Factorizations of Operators

A general problem: Given two operators A and B, when is "B a right-factor of A", i.e., when does there exist some Z such that

(*) $\boxed{A = ZB}$

Obvious "easy" case: If B is invertible, then

$$A = AB^{-1}B = ZB,$$

where $Z = AB^{-1}$ (and this is the only possible Z in this case).

Another obvious case: If $A = 0$ then we can always take $Z = 0$, independently of what B is.

More generally: We have an obvious necessary condition:

$$Bx = 0 \Rightarrow Ax, \quad \text{i.e.,}$$

$$\boxed{\mathcal{N}(B) \subset \mathcal{N}(A)} \quad (= \text{necessary condition}).$$

Second Problem: Can we take $\|Z\| \le 1$ in (*)?

This actually makes the problem easier!

We need a lemma to solve this problem.

12.36 Lemma. If $A \ge 0$, then $B^*AB \ge 0$

Proof: $(B^*ABx, x) = (ABx, Bx) = (Ay, y) \ge 0$, (where $y = Bx$). $\square$