<u>Proof:</u> <u>Step 1.</u> We may always **assume**, without loss of generality, that 0 <u>is</u> <u>not</u> an <u>eigenvalue</u> of RSL.

<u>Prf 1:</u> If 0 <u>is</u> an eigenvalue, then we create a <u>new</u> problem $(RSL)_\mu$ which **does not** have **zero** as an eigenvalue. Done as follows: Pick <u>some</u> μ which is <u>not</u> an eigenvalue (possible by Thm 9.10). Consider the problem

$$(RSL)_\mu : \quad (pf')' + (\overbrace{q+\mu\rho}^{q_\mu} + \lambda\rho) f = 0,$$

same bdry cond. as before.

Then λ is an eigenvalue of $(RSL)_\mu$ $\Longleftrightarrow$ $\lambda+\mu$ is an eigenvalue of RSL, so $\lambda = 0$ is <u>not</u> an eigenvalue of $(RSL)_\mu$. If the **theorem** is true for $(RSL)_\mu$, then we get the **conclusion** of **Thm** 11.1' with λ_n replaced by $\lambda_n - \mu$. Thus, $\lambda_n - \mu$ is **real** and simple ($\Longleftrightarrow$ λ_n real & simple), $|\lambda_n - \mu| \to \infty$ as $n \to \infty$ ($\Longleftrightarrow$ $|\lambda_n| \to \infty$ as $n \to \infty$), and φ_j is a complete orthogonal system.

In the sequel we assume that 0 <u>is</u> <u>not</u> <u>an</u> **eigenvalue**.

<u>Step 2.</u> Case $\boxed{\rho(x) \equiv 1}$. This is the case studied in Thm 10.7 if **we** replace $\lambda \to -\lambda$. By that thm, λ_n is an **eigenvalue** of (RSL) $\Longleftrightarrow$ $-\frac{1}{\lambda_n}$ is an eigenvalue of the operator K, and the eigenfunctions **are** the **same**. Moreover 0 is <u>not</u> an eigenvalue of K. By **Cor.** 8.16, the **set of** eigenvectors of K is <u>complete</u> and <u>orthonormal</u>, and the eigenvalues μ_n of K are <u>real</u> and $\mu_n \to 0$ as $n \to \infty$. Then, $\lambda_n = -1/\mu_n$ are also real, and $|\lambda_n| \to \infty$ as $n \to \infty$.

<u>Note:</u> <u>Usually</u> all but <u>finitely</u> many μ_n satisfy $\mu_n < 0$, and $\boxed{\lambda_n \to +\infty}$ as $n \to \infty$. This is the **reason** why we take $\rho(x) > 0$ instead of $\rho(x) < 0$.

<u>Step 3.</u> general $\boxed{\rho(x) > 0}$ on $[a, b]$.
Let M be the multiplication operator $M\phi = \sqrt{\rho}\,\phi$. Then $M = M^*$ (Ex 7.14a, p.). To clear our thoughts, let us argue <u>formally</u> for a moment: λ eigenvalue of RSL

$$\Longleftrightarrow \quad L\varphi = -\lambda\rho\varphi$$

$$\Longleftrightarrow \quad -\tfrac{1}{\lambda}\varphi = K\rho\varphi$$

$$\Longleftrightarrow \quad -\tfrac{1}{\lambda}\underbrace{\sqrt{\rho}\,\varphi}_{\psi} = \underbrace{\sqrt{\rho}\,K\,\sqrt{\rho}}_{MKM}\underbrace{(\sqrt{\rho}\,\varphi)}_{\psi}$$

$\Longleftrightarrow$ $-\frac{1}{\lambda}$ is an eigenvalue of MKM with eigenfunction $\psi = \sqrt{\rho}\,\varphi = M\varphi$.

We make this <u>precise</u> as follows:

$$(MKM)^* = M^* K^* M^* = MKM, \quad \text{so}$$

MKM is <u>self-adjoint</u> and <u>compact</u> (see homework) $\Rightarrow$ it has a complete set of eigenfunctions ψ_n and corresponding eigenvalues μ_n. Let $\varphi_n = \frac{1}{\sqrt{\rho}}\psi_n$. Then $\sqrt{\rho}\,\varphi_n$ is a complete ortho**normal** set. Moreover, μ_n are real, and $\mu_n \to 0$ as $n \to \infty$, so if we define $\lambda_n = -\frac{1}{\mu_n}$, then λ_n is real, and $|\lambda_n| \to \infty$ as $n \to \infty$. Furthermore:

$$MKM\,\psi_n = \mu_n \psi_n, \quad \text{i.e.,} \quad (\varphi_n = \psi_n/\sqrt{\rho})$$

$$K(\rho\varphi_n) = \mu_n \varphi_n.$$

We know that Kg is continuous for <u>all</u> $g \in C$; so $K(\rho\varphi_n)$ is continuous $\Rightarrow$ φ_n is continuous (μ_n must be $\neq 0$, because otherwise $\rho\varphi_n$ is an eigenfunction of K **with** eigenvalue 0).
By Thm 10.5, the equation $Lf = \rho\varphi_n$ has the <u>unique</u> solution $f = K(\rho\varphi_n) = \mu_n\varphi_n$. Thus, $L(\mu_n\varphi_n) = \rho\varphi_n$, so

$$L\varphi_n = -\left(-\frac{1}{\mu_n}\right)\rho\varphi_j = -\lambda_n\rho\varphi_j$$

Thus, λ_n is an <u>**eigenvalue**</u> of **the** (RSL), with eigenfunction φ_j.

Step 4. Eigenvalues are simple. Suppose that we have two eigenfunctions f and g with the same eigenvalue λ. Then

$$\begin{cases} \alpha_0 f(a) + \alpha_1 f'(a) = 0 \\ \alpha_0 g(a) + \alpha_1 g'(a) = 0 \end{cases}$$

$$\Rightarrow \begin{vmatrix} f(a) & f'(a) \\ g(a) & g'(a) \end{vmatrix} = 0 \Leftrightarrow f(a)g'(a) = g(a)f'(a). \quad (*)$$

Suppose, e.g., that $\alpha_0 \neq 0$ (the case $\alpha_1 \neq 0$ is similar)

Then $f(a) = -\frac{\alpha_1}{\alpha_0} f'(a)$; $g(a) = -\frac{\alpha_1}{\alpha_0} g'(a)$.

Thus, if $f'(a) = 0$ then also $f(a) = 0$, and if $g'(a) = 0$ then also $g(a) = 0$.

We cannot have $f(a) = f'(a) = 0$ or $g(a) = g'(a) = 0$

(**) (neither is identically zero, and the unique sol. of $(pf')' + (q+\lambda\rho)f = 0$ with initial cond $f(a) = 0$, $f'(a) = 0$ is zero). Thus, $f'(a) \neq 0$ and $g'(a) \neq 0$. Put $v(x) =$

$$v(x) = g'(a)f(x) - g(x)f'(a).$$

Then:

i) v satisfies $(pv')' + (q+\lambda\rho)v = 0$

ii) $v(a) = 0$ (because of $(*)$) and $v'(a) = 0$

Thus v is $\equiv 0$ (see (**)). This show that f and g are linearly dependent, so the eigenvalue is simple. $\square$

Note: Step 3 used the assumption $\rho(x) > 0$ for $x \in [a,b]$ (we divided by $\sqrt{\rho}$ at one point). It also used the assumption $p(x) > 0$ for $x \in [a,b]$. Thus, it tells us nothing about singular problems.

Exception: Step 4 applies whenever one of the end-points of (a,b) is regular (it does not use the end-point b).

XI.2. Diagonalization of the S-L-system.

(This section is not found in the book).

Suppose that we want to solve the regular problem

$$\begin{cases} (pf')' + qf = g \\ \alpha_0 f(a) + \alpha_1 f'(a) = 0 \\ \beta_0 f(b) + \beta_1 f'(b) = 0 \end{cases}$$

As before, we write this as

$$\mathcal{L}(f) = g \ ; \quad f \in \mathrm{Dom}(\mathcal{L}).$$

If 0 is not an eigenvalue, and if g is continuous, then the solution is given by

$$f = Kg$$

As in the proof of Thm. 11.1 (page 116) we write this as

$$Mf = MKM(M^{-1}g)$$

Use the spectral theorem 8.15 for MKM:

$$Mf = MKM(M^{-1}g) = \sum_{n=1}^{\infty} \mu_n (M^{-1}g, \psi_n)\psi_n,$$

where $\psi_n = M\varphi_n = \sqrt{\rho}\,\varphi_n$ and

$$(M^{-1}g, \psi_n) = \int_a^b 1/\sqrt{\rho}\, g(x) \sqrt{\rho}\, \overline{\varphi_n(x)}\, dx$$

$$= \int_a^b g(x)\overline{\varphi_n(x)}\, dx = (g, \varphi_n).$$

Denote $\lambda_n = -1/\mu_n$ (= the eigenvalues of the S-L-problem). Then

$$(**) \quad \boxed{f = \sum_{n=1}^{\infty} -\frac{1}{\lambda_n}(g, \varphi_n)\varphi_n}.$$

We write this in matrix form as follows: note

$$f = \sum_{n=1}^{\infty} f_n \varphi_n, \quad g = \sum_{n=1}^{\infty} g_n \rho \varphi_n.$$

Multiply the first equation by $\rho\overline{\varphi_n}$, the second by $\overline{\varphi_n}$, integrate over (a,b), and recall that $\sqrt{\rho}\,\varphi_n$ is orthogonal to $\sqrt{\rho}\,\varphi_m$ for $n \neq m$: (we also suppose that we have normalized φ_n so that $\int_a^b \rho(x)|\varphi_n(x)|^2 dx = 1$):

$$f_n = \int_a^b \rho(x) f(x) \overline{\varphi_n(x)}\, dx$$

$$g_n = \int_a^b g(x) \overline{\varphi_n(x)}\, dx$$

and by (**): $f_n = -\frac{1}{\lambda_n} g_n$, i.e.,

(***) $\lambda_n f_n + g_n = 0.$

Thus, we have the following theorem:

Thm: Suppose that 0 is not an eigenvalue of the regular S-L-system

$$\begin{cases} (pf')' + (q+\lambda\rho) f = 0. \\ \alpha_0 f(a) + \alpha_1 f'(a) = 0, \\ \beta_0 f(b) + \beta_1 f'(b) = 0. \end{cases}$$

Then the solution of the equation (*) on page 118 is given by

$$f = \sum_{n=1}^{\infty} f_n \varphi_n,$$

where $f_n = -\frac{1}{\lambda_n} \int_a^b g(x) \overline{\varphi_n(x)}\, dx$, and $\{\varphi_n\}_{n=1}^{\infty}$ is a complete set of eigenfunctions of the S-L-system with corresponding eigenvalues λ_n. Here we have normalized φ_n so that

$$\int_a^b \rho(x)|\varphi_n(x)|^2 dx = 1$$

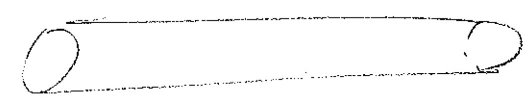
for all n. (Convergence in L^2 with weight ρ). see (***)

Note: If 0 is an eigenvalue, then we must have $g_0 = \int_a^b g(x)\overline{\varphi_0(x)}\, dx = 0$!

XI.3 Separation of variables

We give an example on how this can be used to solve certain partial differential equations. This is not the same example as in the book (which is singular).

Ex: Heat conduction in a bar.



Temperature at point $x \in [0,L]$ at time t is $u(x,t)$. The heat equation says:

(11.1) $\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$ $\quad 0 \le x \le L;\ t \ge 0.$

(11.2) $u_x(0,t) = u_x(L,t) = 0$ (both ends are isolated, no heat enters or leaves the bar)

(11.3) $u(x,0) = h(x)$ (temperature at time $t=0$).

There is a standard technique which applies in this case: Separation of variables: Try to find a solution of the type

$$u(x,t) = f(x)T(t).$$

Then $\frac{\partial u}{\partial t} = f(x)T'(t)$; $\frac{\partial^2 u}{\partial x^2} = f''(x)T(t)$, so (11.1) gives

$$f(x)T'(t) = k f''(x) T(t).$$

Temperature is always >0 ($u(x,t)=0$ is the absolute zero, and heat is always above this), so $k f(x)T(t) > 0$: Divide by this $\Rightarrow$