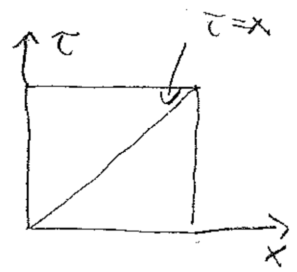
$$\phi(x) = \int_0^x \tau(x-1)\,g(\tau)\,d\tau + \int_x^1 x(\tau-1)\,g(\tau)\,d\tau$$
$$= \int_0^1 k(x,\tau)\,g(\tau)\,d\tau,$$

where

$$k(x,\tau) = \begin{cases} \tau(x-1), & 0 \le \tau \le x \\ x(\tau-1), & x < \tau \le 1. \end{cases}$$



Note: This function is continuous in the square $0 \le x,\tau \le 1$, but its first partial derivatives have a discontinuity jump at the diagonal $x=\tau$:

$\frac{\partial}{\partial\tau}(x(\tau-1)) - \frac{\partial}{\partial\tau}(\tau(x-1)) = x - x + 1 = 1$, and

$\frac{\partial}{\partial x}(x(\tau-1)) - \frac{\partial}{\partial x}(\tau(x-1)) = \tau - 1 + \tau = -1.$

X.2 Variation of parameters formula

The solution of the general S-L-system is similar, but it requires more machinery from the theory of differential equations. The idea is the following:

Suppose that u and v are two linearly independent (not multiples of each other) solutions of the homogeneous equation

$$(p\phi')' + q\phi = 0 \qquad \text{(no boundary values)}$$

Then the general solution of this eq is $Au + Bv$, where A and B are constants. To get a solution of the inhomogeneous equation

$$(\ast)\quad (p\phi')' + q\phi = g$$

we "vary the parameters" A and B, i.e., we replace them by functions $\varphi(x)$ and $\psi(x)$, and try to choose $\varphi(x)$ and $\psi(x)$ in a clever way.

Clever means: We have two unknown functions $\varphi(x)$ and $\psi(x)$ but only one equation $(\ast)$, so we need another equation which we choose in such a way that the computations simplify as much as possible.

Differentiate $\phi = \varphi(x)\,u(x) + \psi(x)\,v(x)$ to get

$$\phi' = \varphi' u + \psi' v + \varphi u' + \psi v'$$
$$\phi'' = (\varphi' u + \psi' v)' + \varphi' u' + \psi' v' + \varphi u'' + \psi v''$$

$$\begin{aligned}(p\phi')' + q\phi = {} & p'(\varphi' u + \psi' v) + p(\varphi' u + \psi' v)' \\ & + \varphi\,[p u'' + p' u' + q u] \\ & + \psi\,\underbrace{[p v'' + p' v' + q v]}_{=0,\ \text{since } u \text{ and } v \text{ solutions}} \\ & + \varphi' p u' + \psi' p v'. \end{aligned}$$

$$= p'(\varphi' u + \psi' v) + p(\varphi' u + \psi' v)' + \varphi' p u' + \psi' p v'.$$

To simplify this as much as possible we choose our second equation to be

$$\varphi' u + \psi' v = 0, \quad \text{and } (\ast) \text{ then becomes}$$
$$p(\varphi' u' + \psi' v') = g$$

Eliminate φ' to get

$$\psi'\, p\,(u v' - u' v) = u g$$

By Lagrange identity, $[p(uv' - u'v)]' = 0$ (the proof of this identity did not use the S-L-boundary conditions), so

$$p(uv' - u'v) = c = \text{a constant.} \Rightarrow$$

$$c\psi' = ug, \quad \text{and a similar}$$

computation gives

$$c\varphi' = -vg.$$

If $c \ne 0$, then we can divide by c and integrate to get

$$\left.\begin{aligned}\varphi(x) &= \tfrac{1}{c}\Big(\int_x^b v(\tau)\,g(\tau)\,d\tau + A\Big)\\ \psi(x) &= \tfrac{1}{c}\Big(\int_a^x u(\tau)\,g(\tau)\,d\tau + B\Big).\end{aligned}\right\}\ (*)$$

Summarizing:

<u>10.1 Lemma.</u> Let u and v be two solutions of the differential equation $(p\phi')' + q\phi = 0$. If $g \in C[a,b]$ and if the constant $c = p(uv' - u'v)$ is not zero, then $\phi = \varphi u + \psi v$ is a solution of the nonhomogeneous equation $(pf')' + q\phi = g$, with φ and ψ given by $(*)$, where A and B are arbitrary constants.

To get further we need to borrow a result from the course in differential equations:

<u>10.2 Existence thm</u> Let P, Q, R be continuous real functions on $[a,b]$, let $x_0 \in [a,b]$, and $y_0, y_1 \in \mathbb{R}$. Then the differential equation
$$y''(x) + P(x)y'(x) + Q(x)y(x) = R(x)$$
has a unique solution satisfying $y(x_0) = y_0$, $y'(x_0) = y_1$.

<u>Proof:</u> Course in differential equations.

This helps us resolve the condition $c \neq 0$ in Lemma 10.1:

<u>Lemma 10.4.</u> The constant c in Lemma 10.1 is nonzero $\Leftrightarrow$ u and v are linearly independent.

<u>Proof.</u> Obviously, if u and v are linearly dependent (one is a multiple of the other), then $c = 0$.

Conversely, suppose that $c = 0$. Since $p(x) > 0$ for all $x \in (a,b)$ we must have $u(x)v'(x) - u'(x)v(x) = 0$ for all $x \in (a,b)$. Fix some $x_0 \in (a,b)$ where $u(x_0) \neq 0$ (if no such point exists then $u(x) \equiv 0$, hence u and v are dependent). Define $\lambda = \frac{v(x_0)}{u(x_0)}$, and

$$y(x) = \lambda u(x) - v(x). \quad \text{Then } y(x_0) = 0 \text{ and}$$
$$y'(x_0) = \lambda u'(x_0) - v'(x_0) = \lambda u'(x_0) - \frac{u'(x_0)v(x_0)}{u(x_0)} = 0,$$
and y is a solution of the equation

$$(*)\quad (py')' + qy = 0,$$

which can be written in the form

$$(**)\quad y'' + \frac{1}{p}\left[p'y' + qy\right] = 0.$$

Since $p(x) \neq 0$ in (a,b), in every subinterval $[c,d]$ of (a,b), we can apply Thm 10.2 to conclude that $(*)$ has a <u>unique</u> solution y on $[c,d]$ satisfying $y(x_0) = y'(x_0) = 0$. However, <u>one</u> such solution is obviously $y(x) \equiv 0$, and by uniqueness we must have $y(x) \equiv 0$. Thus, $\lambda u(x) - v(x) = 0$, for all $x \in [c,d]$. This being true for <u>every</u> subinterval $[c,d]$ of (a,b) we get $\lambda u(x) - v(x) = 0$ on (a,b), so u and v are linearly dependent. □

<u>X, 3 Green's function</u>

In the case of a <u>regular</u> S-L system we can apply Thm 10.2 on the <u>whole</u> interval $[a,b]$ (see $(**)$ above), and we can take $x_0 = a$ or $x_0 = b$. By choosing $x_0 = a$ and y_0 and y_1 **suitably** (so that $\alpha_0 y_0 + \alpha_1 y_1 = 0$) we can choose u in Lemma 10.1 to satisfy
$$\alpha_0 u(a) + \alpha_1 u'(a) = 0,$$
and in the same way we can choose v to satisfy
$$\beta_0 v(b) + \beta_1 v'(b) = 0$$
(under suitable assumptions these will be linearly independent). We also take $A = 0$ and $B = 0$ in formula $(*)$ on p. 95. Then
$$\phi = \varphi u + \psi v$$
$$\phi' = \underbrace{\varphi' u + \psi' v}_{=0} + \varphi u' + \psi v', \quad \text{and}$$

$$\alpha_0 f(a) + \alpha_1 f'(a)$$

$$= \varphi(a)\underbrace{(\alpha_0 u(a) + \alpha_1 u'(a))}_{=0} + \overset{=0}{\psi(a)}\,(\alpha_0 v(a) + \alpha_1 v'(a))$$

$= 0$, and in the same way

$$\beta_0 f(b) + \beta_1 f'(b) = 0.$$

Thus, f satisfies both the boundary conditions if we choose φ and ψ in this way.

10-3 Thm Suppose that 0 is not an eigenvalue of the regular S-L system. Then, for every $g \in C[a,b]$, the inhomogeneous boundary value problem

$$\begin{cases} (pf')' + qf = g, \\ \alpha_0 f(a) + \alpha_1 f'(a) = 0 \\ \beta_0 f(b) + \beta_1 f'(b) = 0 \end{cases}$$

has the unique solution

$$\boxed{f(x) = \int_a^b k(x,\tau)\, g(\tau)\, d\tau,}$$

where

$$\boxed{k(x,\tau) = \begin{cases} \frac{1}{c}\, v(x)\, u(\tau), & a \le \tau \le x \le b \\ \frac{1}{c}\, u(x)\, v(\tau), & a \le x < \tau \le b, \end{cases}}$$

where u and v are two nontrivial solutions of $(pf')' + qf = 0$ satisfying

$$\alpha_0 u(a) + \alpha_1 u'(a) = 0,$$
$$\beta_0 v(b) + \beta_1 v'(b) = 0,$$

and c is the nonzero constant $c = p(uv' - u'v)$.

Proof: Let's first show that $\boxed{c \neq 0.}$ $\boxed{\text{If } c = 0,}$ then by Lemma 10.4, u and v are linearly dependent: $\lambda u + \mu v = 0$. Since neither is $\equiv 0$, we must have both $\lambda \neq 0$ and $\mu \neq 0$. We know that $\alpha_0 u(a) + \alpha_1 u'(a) = 0$, and that $u = -\frac{\mu}{\lambda} v$, so $\beta_0 u(b) + \beta_1 u'(b) = -\mu/\lambda\,(\beta_0 v(b) + \beta_1 v'(b)) = 0.$

Thus, u is an eigenfunction corresponding to the eigenvalue $\lambda = 0$ of the S-L-system. By the assumption, 0 is not an eigenvalue. This contradiction shows that u and v cannot be linearly dependent, i.e., $c \neq 0$.

Next: Uniqueness of f: If there are two solutions f_1 and f_2, then their difference is a nonzero solution of the homogeneous equation (where $g \equiv 0$). Thus, the difference is an eigenfunction of the S-L-system with eigenvalue zero. Impossible!

Next: Is f a solution? By the computation just before Thm 10.3, one solution of

$$(*)\begin{cases} (pf')' + qf = g \\ \alpha_0 f(a) + \alpha_1 f'(a) = 0 \\ \beta_0 f(b) + \beta_1 f'(b) = 0 \end{cases}$$

is $\boxed{f = \varphi u + \psi v,}$ where φ and ψ are given by formula (*) on p. 105 with $A = B = 0$, i.e.,

$$\boxed{\begin{aligned} \varphi(x) &= \frac{1}{c}\int_x^b v(\tau)\, g(\tau)\, d\tau, \\ \psi(x) &= \frac{1}{c}\int_a^x u(\tau)\, g(\tau)\, d\tau \end{aligned}}$$

Multiplying the first formula by $u(x)$ and the second by $v(x)$, and adding the results we get

$$f(x) = \frac{1}{c}\int_a^b k(x,\tau)\, g(\tau)\, d\tau,$$

where $k(x,\tau)$ is the function given in the theorem. The only thing that we have been careless about is how "smooth" f is. The theory says that f is continuously differentiable, and that $f'' \in L^2(a,b)$. However, from (*) we see that

$$f''(x) = \frac{1}{p(x)}\left[g(x) - p'(x) f'(x)\right], \quad \text{so}$$

f'' is actually continuous, i.e., f is two times cont. differentiable. □

X.4 The Inverse of L is Compact

10.5 Thm Suppose that 0 is not an eigenvalue of the regular S-L-problem. Let K be the integral operator

$$(Kg)(x) = \int_a^b k(x,\tau)\, g(\tau)\, d\tau,$$

where k is the Green's function given in Theorem 10.3:

$$\boxed{k(x,\tau) = \begin{cases} \frac{1}{c} v(x)\, u(\tau), & a \le \tau \le x \le b, \\ \frac{1}{c} u(x)\, v(\tau), & a \le x < \tau \le b, \end{cases}}$$

where u and v are two nontrivial solutions of the homogeneous equation $(p\phi')' + q\phi = 0$ satisfying

$$\alpha_0 u(a) + \alpha_1 u'(a) = 0,$$
$$\beta_0 v(b) + \beta_1 v'(b) = 0.$$

Then K is a compact self-adjoint operator (even Hilbert-Schmidt), and the solution of the nonhomogeneous boundary value problem

$$\boxed{L\phi = g, \quad \phi \in \mathcal{D}(L)}$$

is given by $\boxed{\phi = Kg}$ (whenever g is continuous).

Proof: The function $k(x,\tau)$ is continuous (both formulas give the same value $k(x,x) = \frac{1}{c} v(x)\, u(x)$ when we take $\tau = x$) on $[a,b] \times [a,b]$, hence bounded, and so K is a Hilbert-Schmidt operator (see Ex. 7.26 on p. and Thm. 8.8 on p. 89), hence compact (by Thm 8.7). It is self-adjoint by Ex. 7.14b, p. . The rest follows from Thm 10.3. □

General comment: There is one **sad** limitation in Theorem 10.5: we had to assume that g is continuous, because we wanted to use standard results from the course on differential equations. We know that Kg is well-defined for all $g \in L^2(a,b)$, and not just for continuous g, but if g is not continuous, eq. how do we prove that the non-homogeneous $L\phi = g$, $\phi \in \mathcal{D}(L)$ still has a solution ϕ, and that this solution is still given by $\phi = Kg$?

Solution 1: Go through the proofs in the differential equations course, and **replace** "ϕ two times continuously differentiable" by "ϕ continuously differentiable, and $\phi'' \in L^2(a,b)$". This requires not only Lebesgue integral theory, but also the theory of how to differentiate functions which are differentiable only almost everywhere, and to show that a "function is the integral of its derivative". It needs Too heavy machinery (but it gives a **beautiful** theory when you are done!). (Everything fits perfectly into place) (Not more difficult **than** what we **have** done, but it **takes a** lot of work).

Solution 2. We **cheat** and prove only those results which **we** actually need to proceed. This is the one **we** take.

10.6 Lemma. Suppose that 0 is **not** an eigenvalue of the **regular** S-L-system, and let K be the operator in Thm. 10.5. Then, for every $g \in L^2(a,b)$, Kg is (continuously) differentiable, and

(*) $\quad (Kg)' = \varphi u' + \psi v'$

(with u, v, φ, ψ as in that theorem and its proof).

Proof: We have (see p. 108) $\phi = \varphi u + \psi v$, where u and v are cont. differentiable, but φ and ψ need **not be differentiable** everywhere:

$$\left.\begin{aligned} \varphi(x) &= \frac{1}{c}\int_x^b v(\tau)\,g(\tau)\,d\tau \\ \psi(x) &= \frac{1}{c}\int_a^x u(\tau)\,g(\tau)\,d\tau \end{aligned}\right\} \text{ where } g \in L^2(a,b).$$

If g is **continuous**, then φ and ψ are **differentiable**, and so is Kg. In this **case**

$$\begin{aligned}(Kg)' &= \varphi' u + \varphi u' + \psi' v + \psi v' \\ &= -\frac{vgu}{c} + \varphi u' + \frac{ugv}{c} + \psi v' \\ &= \varphi u' + \psi v'. \end{aligned}$$

Fortunately, this **formula** does not contain φ' or ψ' (which **need not always exist**).

The right hand side of (*) is **continuous** (in all cases). Therefore (*) is equivalent to:

$$(**) \quad (Kg)(x) - (Kg)(a) = \int_a^x (\varphi u' + \psi v')(\tau)\,d\tau, \quad a \le x \le b$$

(we get this from (*) by **integrating**, and **we** get (*) from (**) by **differentiating**). This is equal to

$$(Kg)(x) = \underbrace{u(a)\,\frac{1}{c}\int_a^b v(\tau)g(\tau)\,d\tau}_{(Mg)(x)} + \underbrace{\int_a^x (\varphi u' + \psi v')(\tau)\,d\tau}_{(Ng)(x).}, \quad a \le x \le b.$$

We know that $Kg = Mg + Ng$ if g is continuous (we saw at the beginning of the proof that (*) holds when g is continuous). The operator M is continuous from $L^2(a,b)$ to $L^2(a,b)$, and every function in its range is a constant function (independent of x). It is

actually continuous **from** $L^2(a,b)$ with the usual inner-product **norm** to $C[a,b]$ with the sup-norm. Also the operator N is continuous from $L^2(a,b)$ with the inner prod. norm to $C[a,b]$ with the sup-norm (use Cauchy-Schwartz; see the book).

If we have an arbitrary $g \in L^2(a,b)$, then we take a sequence $g_n \in C[a,b]$, so that $g_n \to g$ in $L^2(a,b)$. Then

$$Kg_n = Mg_n + Ng_n \longrightarrow Mg + Ng \text{ in } C[a,b]$$

$$\longrightarrow Kg \text{ in } C[a,b].$$

(since also the integral operator K is continuous from $L^2(a,b)$ with the inner prod. norm to $C[a,b]$ with the sup-norm).

Therefore $Kg = Mg + Ng$ for all $g \in L^2(a,b)$, i.e., (**) holds. Differentiating (**) we get (*). □

10.7 Thm Suppose that 0 is not an eigenvalue of the regular S-L-system, and let K be the integral oper. in Thm. 10.5. Then

(i) 0 is not an eigenvalue of K,

(ii) λ is an eigenvalue of K
$\iff$ $1/\lambda$ is an eigenvalue of L

(iii) The corresponding eigenvectors of K and L in (ii) are the same.

Note: In this theorem we formally take $O(x) \equiv -1$ in the L-S-system

Proof. (i) Suppose that $Kg = 0$. Then $\phi = \varphi u + \psi v = 0$ (see p. 108). Since $Kg = 0$, also $(Kg)' = 0$, so by lemma 10.6,

$$\begin{cases} \varphi u + \psi v = 0 & \big|\; v' \\ \varphi u' + \psi v' = 0 & \big|\; -v \end{cases} \Rightarrow \varphi u v' - \varphi u' v = 0$$

$\Rightarrow \quad \underbrace{\varphi(uv'-u'v)}_{= C/p(x)} = 0$, where C is the constant in Thm 10.3.

$\Rightarrow \quad \varphi \equiv 0.$

A similar computation gives $\psi \equiv 0$. Therefore, by the definition of φ and ψ,

$$\left.\begin{aligned} \int_x^b v(\tau) g(\tau)\, d\tau &= 0 \\ \int_a^x u(\tau) g(\tau)\, d\tau &= 0 \end{aligned}\right\} \quad a \le x \le b$$

Thus, vg and ug are "null functions" ($= 0$ a.e.) in $L^2(a,b)$. Since $p(x)\,[u(x)v'(x) - u'(x)v(x)] = C \neq 0$, u and v **cannot** both **vanish at the same** point x. Thus, g is a "null function". This shows that K is one-to-one, i.e., 0 is not an eigenvalue of K.

(ii)-(iii) **Suppose** $\phi \in D(L)$ and $\boxed{L\phi = \lambda\phi}$, $\phi \neq 0$. (I.e., ϕ is an eigenvector of L with eigenvalue λ). Then $\lambda \neq 0$ (see the assumption) and $\lambda\phi$ is continuous. Put $g = \lambda f$ **in Thm** 10.5 to get

$$\phi = KL\phi = K(\lambda\phi) = \lambda K\phi \iff$$
$$Kf = \tfrac{1}{\lambda}\phi.$$

Thus, every eigenvector ϕ of L with eigenvalue λ is also an eigenvector of K with eigenvalue $1/\lambda$.

If instead g is an eigenvector of K with eigenvalue μ: $Kg = \mu g$. By (i), $\mu \neq 0$, and by lemma 10.6, $g = \frac{1}{\mu}Kg$ is continuous. By Thm 10.5, the solution of $L\phi = g$ is given by $\phi = Kg$. Thus $LKg = g$, so $L(\mu g) = g$, i.e., $Lg = \frac{1}{\mu}g$. Therefore g is an eigenvector of the S-L-problem with eigenvalue $1/\mu$.

XI. Eigenfunction expansions

Idea: We use the eigenvalues and eigenfunctions of L (or $L^{-1} = K$) to diagonalize the operator L, and to get a simple series **expansion** for the **solution** of the S-L-problem

$$L\phi = g\,; \quad \phi \in \mathrm{Dom}(L).$$

XI.1. Simple eigenvalues

We recall the regular Sturm-Liouville problem

$$(RSL) \begin{cases} (pf')' + (q + \lambda\rho) f = 0 & a \le x \le b \\ \alpha_0 \phi(a) + \alpha_1 \phi'(a) = 0 \\ \beta_0 \phi(b) + \beta_1 \phi'(b) = 0 \end{cases}$$

11.0 Defn. An eigenvalue λ of (RSL) is simple if any two solutions u and v of (RSL) are linearly dependent (one is a multiple of the other).

Note: In all examples which we have computed so far the eigenvalues were simple!

11.1 Sturm-Liouville Thm. The regular problem (RSL) has ∞ many (different) eigenvalues, each eigenvalue λ_n is real and simple, and $|\lambda_n| \to \infty$ as $n \to \infty$. The corresponding eigenfunctions φ_j are such that $\{\sqrt{\rho}\,\varphi_j\}_{j=1}^{\infty}$ form a complete orthogonal system in $L^2(a,b)$ (and we can make it orthonormal by scaling φ_j so that $\|\varphi_j\| = 1$).