Hilbert Spaces I-II, 5+5 sp

Aim: A Hilbert space is the infinite-dimensional version of an n-dimensional euclidean space, and it is one of the most used mathematical tools in applied mathematics, stochastics, physics, control theory, etc. For example, the modern theory of partial differential equations is largely based on Hilbert space techniques, as is standard optimization theory (including optimal control), wavelet compressions, and many of the standard models in quantum physics. In this course we learn about the properties of operators in Hilbert spaces and possible applications of Hilbert space techniques.

Contents, Part I: Inner product spaces, normed spaces, Hilbert and Banach spaces, orthogonal expansions, classical Fourier series, dual spaces, linear operators.

Contents, Part II: Compact operators, spectrum and eigenvalues, Sturm-Liouville operators, Green's functions, eigenfunction expansions, positive operators, contractive operators, singular values.

Study forms: Lectures and home work.

Litteratur: N. Young, An introduction to Hilbert space, Cambridge University Press, 1988. Lecture notes will also be handed out.

Prerequisits: Analysis and linear algebra.

Time: The two fall semesters 2010, Tue. 15-17, Wed.13-15, and Thu. 10-12 in Hilbertrummet (ASA B329). The first lecture will be given on Wed. **Sept. 1, 13-15.**

273031-2 Hilbert Spaces I–II, 2010

- Aim and Contents: See www home page
- Exam: One final exam at the end of each part of the course. This exam contributes to 50% of the final grade.
- Homework: About 5 problems each week. These are graded (on a very course scale), and they contribute to 50% of the final grade. To get the maximal score for the homework it suffices to solve 75% of the problems.
- Later exams: In later exams the home work points are no longer counted (or just counted partially).
- Course material: See www home page
- Language: English

0 Introduction

"Advanced mathematical methods" have become increasingly popular in many engineering and other applications, and very many of these are based on Hilbert space methods. This applies, in particular, to

- Theoretical and numerical solutions of partial differential equations describing e.g.
  - hydrodynamics
  - airflows (design of vehicles)
  - heat conduction (powerplants)
  - chemical processes (distillers)
  - all sorts of processes which take place in a distributed medium
- Optimization techniques in many fields
- Control theory: Optimal and robust control.
- Population dynamics
- Scattering theory (reconstruction of data, thomography, etc.)
- Data compression (wavelets).

This course contains both the basic theory (60%) and more specialized topics important in applications (40%).

- Signal Processing

General comment: The numbering of theorems etc. follows the numbering in

N. Young, An Introduction to Hilbert Space

I Inner Product spaces

1.1 Defn A complex vector space V (= a vector space over $\mathbb{C}$) ($\mathbb{C}$ = complex plane) is a set, whose elements are called vectors, where it is possible to perform two types of operations:
- Addition of two vectors
- Multiplication of a vector and a scalar.

1) Addition rules:
   a) $x+y = y+x$
   b) $(x+y)+z = x+(y+z)$
   c) There is a zero vector, denoted by 0, so that $x+0 = x \quad \forall x \in V$
   d) Every $x \in V$ has an "antivector", denoted $-x$, so that $x+(-x) = 0$.

2) Multiplication rules
   a) $\lambda(\mu x) = (\lambda\mu)x$
   b) $1 \cdot x = x$

3) Interaction rules
   a) $(\lambda+\mu)x = \lambda x + \mu x$
   b) $\lambda(x+y) = \lambda x + \lambda y$

1.1A. Defn A real vector space V (a vector space over $\mathbb{R}$) obeys the same rules, but we only allow real scalars (in the above rules we take λ and μ to be real).

Note: Every complex vector space is also a real vector space. Every real vector space V has a complexification, which we can identify with $V \times V$. We interpret a point $\begin{bmatrix} x \\ y \end{bmatrix} \in \begin{bmatrix} V \\ V \end{bmatrix}$ as "$x + iy$".

Example: l^2 in the book.

1.2 Defn. Let V be a vector space.

A) A mapping from $V \times V$ to $\mathbb{C}$, which we denote by $x, y \to (x,y)$, is a (complex) semi-inner product if for all $x, y, z \in V$ and all $\lambda \in \mathbb{C}$,

(i) $(x,y) = \overline{(y,x)}$, (complex conjugates)
(ii) $(\lambda x, y) = \lambda (x,y)$,
(iii) $(x+y, z) = (x,z) + (y,z)$,
$(iv)_0$ $(x,x) \geq 0$.

We get an inner product if we strengthen $(iv)_0$ to

(iv) $(x,x) > 0$ whenever $x \neq 0$.

B) An inner product space is a vector space with an inner product. A semi-inner product space is defined analogously.

Note: Real inner product spaces are defined in the same way, with the exception that ~~its~~ values are real. In particular, (i) becomes

$(i)_R$ $(x,y) = (y,x)$ (symmetric)

Note: Complex semi-inner products are sometimes called sesquilinear (sesqui = $1\frac{1}{2}$) forms (linear in first variable, conjugate-linear in second), and they are nonnegative (property $(iv)_0$) and symmetric (property (i)). An inner product is furthermore nondegenerate (property (iv)).

Ex. $(A,B) = \text{trace}(B^* A)$ (Homework).

1.6 Defn. We call $\|x\| = \sqrt{(x,x)}$ a semi-norm if V is a semi-inner product space. It is a norm if V is an inner product space.

1.9 Cauchy-Schwarz inequality

$$|(x,y)| \leq \|x\| \, \|y\|.$$

True both for semi-inner and inner products. In the case of a inner product we have equality only when x and y are linearly dependent ($x = \lambda y$ or $y = \lambda x$ for some $\lambda \neq 0$).

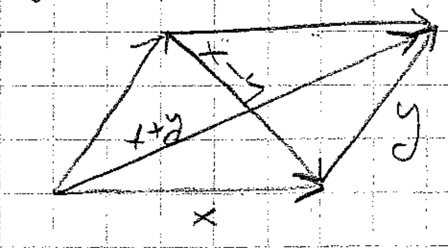
1.11 Triangle inequality

$$\|x+y\| \leq \|x\| + \|y\|$$

(true both for semi-inner and inner products).

1.13 Parallellogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$



Sum of squares of diagonal = sum of squares of sides.

1.14 Polarization identity (complex case)

$$4(x,y) = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2$$

(you can recover inner product from norm)

1.14 A Polarization identity (real case)

$$4(x,y) = \|x+y\|^2 - \|x-y\|^2$$

1.16 Ex. RL^2 = all rational functions whose poles do not lie on the unit circle (enhetscirkeln)

$$\partial D = \{z \in \mathbb{C} \mid |z| = 1\}.$$

RH^2 = all rational functions whose poles lie outside of the closed unit disk (enhetsskivan)

$$D = \{z \in \mathbb{C} \mid |z| \le 1\}.$$

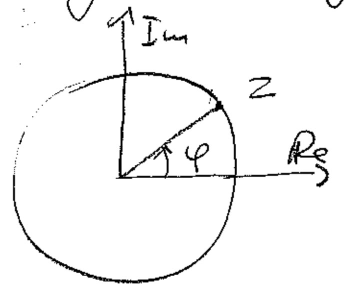
Both are vector spaces, and $RH^2 \subset RL^2$. In both of these, we define the inner product

$$(*) \quad (f,g) = \frac{1}{2\pi i} \oint_{\partial D} f(z)\overline{g(z)}\,\frac{dz}{z} = \frac{1}{2\pi}\int_{\varphi=0}^{2\pi} f(e^{i\varphi})\overline{g(e^{i\varphi})}\cdot d\varphi$$

Claim (= påstående). Both RH^2 and RL^2 are inner product spaces.

Proof. A) It is easy to show that they are vector spaces.

B) Is this an inner product? We begin by rewriting (*) as an ordinary integral.



Every $z \in \partial D$ has $|z| = 1$, and can be written as

$$z = e^{i\varphi}, \quad -\pi \le \varphi < \pi$$

(use polar decomposition).

$$dz = ie^{i\varphi}, \quad \frac{1}{z} = e^{-i\varphi}, \quad z \in \partial D \Leftrightarrow -\pi \le \varphi < \pi,$$

so

$$(f,g) = \frac{1}{2\pi i}\oint_{\partial D} f(z)\overline{g(z)}\,dz$$

$$= \frac{1}{2\pi i}\int_{-\pi}^{\pi} f(e^{i\varphi})\overline{g(e^{i\varphi})}\,e^{-i\varphi}\,ie^{i\varphi}\,d\varphi$$

$$\boxed{(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi})\overline{g(e^{i\varphi})}\,d\varphi}$$

(This is related to Fourier transforms).

Conditions (i) – $(iv)_0$ in Defn. 1.2 are now easily verified (based on ordinary integration theory, no complex function theory needed). Property (iv) uses properties of rational functions: If $(f,f) = 0$, then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\varphi})|^2\,d\varphi = 0,$$

hence $f(z) = 0$ when $|z| = 1$, and we must have $f \equiv 0$.

By using residue calculus from analytic function theory we can compute these inner products exactly (more about this later ???) For example:

Take $\phi = \frac{1}{z-\alpha}$, $g = \frac{1}{z-\beta}$,

where $|\alpha| < 1$, $|\beta| < 1$. Then

$$(\phi, g) = \frac{1}{2\pi i} \int_{\partial D} \frac{1}{z-\alpha} \frac{1}{\bar{z}-\bar{\beta}} \frac{dz}{z}.$$

On the unit circle we have

$|z| = 1 \iff |z|^2 = 1 \iff z\bar{z} = 1 \iff$

$$\boxed{\bar{z} = 1/z},$$

so this can be written as

$$(\phi, g) = \frac{1}{2\pi i} \oint_{\partial D} \frac{1}{z-\alpha} \frac{1}{1/z - \bar{\beta}} \frac{1}{z} \, dz$$

$$= \frac{1}{2\pi i} \oint_{\partial D} \frac{1}{z-\alpha} \frac{1}{1-\bar{\beta} z} \, dz$$

= sum of the residues of the analytic function $\frac{1}{z-\alpha} \frac{1}{1-\bar{\beta} z}$ inside the unit circle. This function has two poles: one at $z=\alpha$ (inside) and $z = 1/\bar{\beta}$ (outside). The residue at the pole $z=\alpha$ is equal to $\frac{1}{1-\bar{\beta} z}$, evaluated at $z=\alpha$, so we get in this case

$$\left(\frac{1}{z-\alpha}, \frac{1}{z-\beta}\right) = \frac{1}{1-\alpha\bar{\beta}}. \qquad \square$$

Note: Residue calculus is not required in exam!

II. Normed spaces (normerade rum)

II.1 Definitions

2.1 Defn. A) A seminorm $\|\cdot\|$ is a function $E \to \mathbb{R}$ (where E is a vector space) with the following properties: For all $x, y \in E$ and all $\lambda \in \mathbb{C}$,

$(N1)_0$	$\|x\| \geq 0$	(nonnegative)
$(N2)$	$\|\lambda x\| = |\lambda| \, \|x\|$	(homogeneous)
$(N3)$	$\|x+y\| \leq \|x\| + \|y\|$	(triangle inequality)

B) This seminorm is a norm if $(N1)_0$ is replaced by the stronger condition

$(N1)$ $\|x\| > 0$ whenever $x \neq 0$.

C) A seminormed space is a vector space with a given seminorm. A normed space is a vector space with a given norm.

By using a norm we define what we mean by a ball:

2.2 Ex. In $\mathbb{R}$ and $\mathbb{C}$ we always use the norm $\|x\| = |x|$ (= absolute value).

II.2 Open and closed sets, continuity

With the help of a norm $\|\cdot\|$ we can define open and closed balls with center $x_0 \in E$ and radius $r > 0$:

Open: $B(x_0, r) = \{x \in E \mid \|x - x_0\| < r\}$

Closed: $\overline{B}(x_0, r) = \{x \in E \mid \|x - x_0\| \leq r\}$.

We then define continuity in the same way as we do in $\mathbb{R}$ or $\mathbb{R}^n$:

> Use the standard analysis-definition, but replace absolute values $|x-y|$ by norms $\|x-y\|$

We also define *open* and *closed* sets in the same way as in $\mathbb{R}$ or $\mathbb{R}^n$, but again we replace all absolute values by norms. Doing so,

$$\boxed{\text{Almost all of the standard "analysis"-results remain valid}}$$

Exeptions: i) Compact $\neq$ closed & bounded

ii) Cauchy-sequences need not converge

For more details, see "Analysis 2" or
Sjöberg: Hilbertrum, or
Sjöberg: Funktionalanalys.

2.4 Thm Seminorms and norms are *continuous* functions, i.e., the mapping $x \mapsto \|x\|$ is always continuous from E to $\mathbb{R}$.

Proof. Define $f(x) = \|x\|$. We claim that f is continuous at every point $x_0 \in E$. Kontinuity at a point $x_0 \in E$ means:

(*) $\forall\, \varepsilon > 0 \;\exists\, \delta > 0$, such that $|f(x) - f(x_0)| < \varepsilon$ for all x satisfying $\|x - x_0\| < \delta$

or equivalently, replace $\|x - x_0\| < \delta$ by $x \in B(x_0, \delta)$.

Write this out: $|f(x) - f(x_0)| = \big|\|x\| - \|x_0\|\big|$.

The proof of (*) is based on the Δ-inequality: we have

$$\|x\| = \|x - x_0 + x_0\| \le \|x - x_0\| + \|x_0\|$$
$$\|x_0\| = \|x_0 - x + x\| \le \|x_0 - x\| + \|x\|$$

$$\Rightarrow \quad \|x\| - \|x_0\| \le \|x - x_0\|$$
$$\|x_0\| - \|x\| \le \|x_0 - x\| = \|x - x_0\|, \quad \text{so}$$

$$\big|\|x\| - \|x_0\|\big| \le \|x - x_0\|.$$

Choosing $\boxed{\delta = \varepsilon}$ we get (*). □

2.5 Thm. Let $\lim_{n\to\infty} x_n = x$ in E (i.e., $\|x_n - x\| \to 0$)

$\lim_{n\to\infty} y_n = y$ in E (i.e., $\|y_n - y\| \to 0$)

$\lim_{n\to\infty} \lambda_n = \lambda$ in $\mathbb{C}$ (i.e., $|\lambda_n - \lambda| \to 0$).

Then $\lim_{n\to\infty} (x_n + y_n) = x + y$

$\lim_{n\to\infty} \lambda_n x_n = \lambda x$.

Thus *addition* and *scalar multiplication* are *continuous operations* in V.

Proof. Clearly

$$\|(x_n + y_n) - (x + y)\| = \|(x_n - x) + (y_n - y)\|$$
$$\le \|x_n - x\| + \|y_n - y\| \to 0 \text{ as } n \to \infty$$

$$\|\lambda_n x_n - \lambda x\| = \|\lambda_n (x_n - x) + (\lambda_n - \lambda) x\|$$
$$\le |\lambda_n| \, \|x_n - x\| + |\lambda_n - \lambda| \, \|x\|$$
$$\to 0 \text{ as } n \to \infty.$$

This means that $\lim_{n\to\infty} x_n + y_n = x + y$ and

$\lim_{n\to\infty} \lambda_n x_n = \lambda x$. □

II.3 Subspaces and the linear span

Defn. We call $F \subset E$ a (linear) subspace of E (where E is a vector space) iff it is closed under addition and scalar multiplication, i.e.,

(i) $x \in F,\ y \in F \Rightarrow x+y \in F$
(ii) $x \in F,\ \lambda \in \mathbb{C} \Rightarrow \lambda x \in F$

2.7. Ex: A subspace which is not closed.
Take $E = RH^2 =$ all rational functions whose poles lie outside of the unit disk (see Ex. 1.16), with the inner product on p. 5.
Take $F =$ all polynomials. Clearly F is a subspace of E. We claim that F is not closed, i.e., if each p_n is a polynomial and $p_n \to p$ in RH^2, then p need not be a polynomial.

Counterexample Take $p(x) = \dfrac{1}{x+2}$.
Then $p \in RH^2$. We know from the theory of analytic functions (or Analysis II) that

$$\frac{1}{x+2} = \frac{1}{2}\cdot\left(\frac{1}{1+\frac{x}{2}}\right) = \frac{1}{2}\left(1-\frac{x}{2}+\left(\frac{x}{2}\right)^2-\cdots\right),$$

where the series converges uniformly in x for example for all $|x| \le 1$, i.e., on the unit disk. Define

$$p_1(x) = \tfrac{1}{2}$$
$$p_2(x) = \tfrac{1}{2}\left(1-\tfrac{x}{2}\right)$$
$$p_3(x) = \tfrac{1}{2}\left(1-\tfrac{x}{2}+\left(\tfrac{x}{2}\right)^2\right) \quad \text{etc.}$$

Then $p_n(x) \to p(x)$ uniformly for $|x| \le 1$, so

$$\|p_n - p\|^2_{RH^2} = (p_n - p, p_n - p)_{RH^2}$$
$$= \int_{\varphi=0}^{2\pi} |p_n(e^{i\varphi}) - p(e^{i\varphi})|^2\, d\varphi \to 0 \quad \text{as } n \to \infty.$$

Thus $p_n \to p$ in E, but $p \notin F$, so F is not closed. □

Defn. We get the closure $\overline{F}$ of F by adding those points $x \in E$ which are limit points of F, i.e.,

$$\boxed{x \in \overline{F} \iff \text{there is a sequence } x^k \in F \text{ such that } x = \lim_{k\to\infty} x^k}$$

2.9 Thm. If F is a subspace of E, then so is $\overline{F}$ (i.e., also $\overline{F}$ is a subspace of E).

Proof. See the book, p. 16. □

2.10 Defn Let $A \subset E$ ($E =$ a normed space). The linear span (lineära höljet) "$[A]$" of A is the intersection of all subspaces of E that contain A. The closed linear span (slutna lineära höljet) "$\mathrm{clin}(A)$" is the intersection of all closed subspaces of E which contain A.

Thus:
$$[A] = \bigcap_{\substack{A \subset F \\ F \text{ subspace of } E}} F$$

$$\mathrm{clin}(A) = \bigcap_{\substack{A \subset F \\ F \text{ closed subspace} \\ \text{of } E}} F.$$

These definitions are very nonintuitive. A more intuitive description will be obtained soon.

2.11 Thm. Both $[A]$ and $\mathrm{clin}(A)$ are subspaces of E and $[A] \subset \mathrm{clin}(A)$.

Proof. i) $[A]$ is a subspace:
Let $x, y \in [A]$, $\lambda \in \mathbb{C}$. We claim that $x+y \in [A]$ and $\lambda x \in [A]$ (this means that $[A]$ is a subspace).
a) To show that $x+y \in [A]$ we must show that if F is an arbitrary subspace of E and $A \subset F$, then $x+y \in F$ (because then it belongs to the intersection). Let F be such a subspace. Then
$x \in F$ (because $x \in [A] \subset F$)
$y \in F$ (because $y \in [A] \subset F$)
$x+y \in F$ (because F is a subspace)
$\lambda x \in F$ (— " —)
Thus, both $x+y \in F$ and $\lambda x \in F$, so these vectors belong to the intersection of all such F, i.e., $x+y \in [A]$ and $\lambda x \in [A]$.

ii) $\mathrm{clin}(A)$ is a subspace: same proof as above
iii) $[A] \subset \mathrm{clin}(A)$ easy. $\square$

By expanding this argument slightly we find that

$[A]$ = the smallest subspace containing A
$\mathrm{clin}(A)$ = the smallest closed subspace containing A.

2.11a Description of $[A]$ A vector $x \in E$ belongs to $[A]$ if and only if it can be written as a (finite) sum

$$x = \sum_{n=1}^{m} \lambda_n a_m, \quad \text{where each } a_m \in A.$$

Proof: See the book.

2.12 Thm $\mathrm{clin}(A)$ is the closure of $[A]$.

Proof: See the book.

Notation: In the sequel we denote
$\mathrm{clin}(A) = \overline{[A]}$ (= closure of $[A]$).

Defn. The vectors $v_1, \dots, v_n$ are linearly independent if there do not exist constants $\lambda_1, \dots, \lambda_n$, where at least one $\lambda_i \neq 0$, such that
$\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n = 0$.

Defn. A vector space V has dimension n $(<\infty)$ if it is possible to find exactly n linearly independent vectors $x_1, x_2, \dots, x_n$, such that every $x \in V$ can be written in the form

$$x = \sum_{i=1}^{n} \lambda_i x_i$$

(for suitably chosen "coordinates" $\lambda_i \in \mathbb{C}$). We call $\{x_i\}_{i=1}^{n}$ a basis of V.

2.13 Thm Every n-dimensional normed (complex) space E is isomorphic to $\mathbb{C}^n$, i.e., it is possible to construct a linear map $G: E \to \mathbb{C}^n$ which maps E one-to-one onto $\mathbb{C}^n$, and both this map G and its inverse G^{-1} are continuous.

Proof. See the book, or "Analysis I." Thus,

Every finite-dimensional normed space has (almost) exactly the same properties as $\mathbb{C}^n$

so we already know "everything" about these spaces.

Only infinite-dimensional spaces provide new and interesting results.

II.4 Real vector spaces

To this point we have mainly been discussing complex vector spaces.

Defn. A real vector space, a real inner product space (also called a Euclidean space) and a real normed space is defined in the same way as the corresponding complex spaces, but we only multiply the vectors by real numbers. The set of numbers that we are multiplying the vectors with are called scalars. Thus, a

- real vector space = a space with real scalars
- (complex) vector space = a space with complex scalars.

In the real case inner products take real values, and the condition

$$(x,y) = \overline{(y,x)} \quad \text{becomes} \quad \boxed{(x,y) = (y,x)}.$$

Note: Most results are true for both real and complex spaces, but sometimes we must use one or the other. For example, the eigenvalues of a real matrix may be complex!

When we compute eigenvalues and eigenvectors we throughout replace $\mathbb{R}^n$ by $\mathbb{C}^n$.