273032 Hilbert Spaces II

Home Work #3, Due 17-11-2010

10.2. Let ω be positive, not an integer multiple of π. Find the Green's function for the boundary value problem

(43)

$$f'' + \omega^2 f = g,$$
$$f'(0) = 0 = f'(1).$$

Double points ?

What happens if we try this with $\omega = 0$?

10.3. Find the Green's function for the inhomogeneous Sturm–Liouville problem

(44)

$$f'' = g,$$
$$f(0) = 0, \qquad f(1) + f'(1) = 0.$$

Hint- Wrong answer in book

10.6. Give an example to show that the differential operator

(45)

$$L: D(L) \to L^2(a, b)$$

corresponding to a regular Sturm–Liouville system, as defined in (9.19), need not be surjective.

Hint- Make $\lambda = 0$ an eigenvalue ?

11.1. By applying the Sturm–Liouville theorem to the system

(46)

$$f'' + \lambda f = 0,$$
$$f(0) = 0 = f'(\pi),$$

show that, for any $g \in L^2(0, \pi)$,

$$g(x) = \sum_{j=1}^{\infty} C_j \sin(j - \tfrac{1}{2})x$$

in the norm of $L^2(0, \pi)$, where

$$C_j = \frac{2}{\pi} \int_0^{\pi} g(x) \sin(j - \tfrac{1}{2})x \, dx, \qquad j \in \mathbb{N}.$$

good physical grounds for this criterion. Indeed, one might wonder why this idea was such a late starter. Part of the answer must be that engineers were unaware of the relevant mathematical theorems and operator theorists of the engineering problem. Credit for making the connection seems to be due to G. Zames and J. W. Helton (see references in Francis, 1986). The theory is now developing rapidly and looks to be a promising new tool.

What happens if we try to add a dash of robustness to the problem of internal stabilization? Suppose we are given a transfer function G, but suspect it may be in error by up to ε. We believe that the 'real' transfer function of the system is $G + \Delta G$, where $G + \Delta G$ has the same number of poles in the closed right half plane as G and

$$\|\Delta G(s)\| < \varepsilon, \qquad \text{all } \operatorname{Re} s > 0$$

(here $\|\cdot\|$ denotes the operator norm on $\mathscr{L}(U, X)$). Let us denote the class of such functions $G + \Delta G$ by $V(G, \varepsilon)$. Since we do not know ΔG we need to find a controller which stabilizes everything in $V(G, \varepsilon)$. It is not hard to show that a rational function K internally stabilizes every function in $V(G, \varepsilon)$ if and only if

(i) K internally stabilizes G, and
(ii) $\|K(I - GK)^{-1}(s)\| \leqslant 1/\varepsilon$, all $\operatorname{Re} s > 0$.

There may or may not exist a K which satisfies these two conditions: it depends on the characteristics of G and the size of ε. To ascertain whether there does we take from (14.5) the parametrization of all K which satisfy (i) and substitute in (ii). By virtue of some happy coincidences the result simplifies better than we are entitled to expect, and we find that

$$K(I - GK)^{-1} = T_1 + T_2 Q T_3$$

where T_1, T_2 and T_3 are polynomial matrices and Q is the free parameter (a rational function analytic and bounded on the right half plane).

In summary, if we are given a transfer function G and a tolerance ε, we can determine whether there exists a controller (representable by a rational matrix function) which internally stabilizes all plants lying within ε of G in the H^∞-norm and having the same number of 'unstable poles' as G provided we can solve a certain concrete problem about analytic matrix functions. That is, we can constructively find polynomial matrices T_1, T_2 and T_3, derived from G, such that the answer to the robust stabilization question is yes or no depending on whether there does or does not exist a bounded analytic matrix function Q in the right half plane such that

$$\|(T_1 + T_2 Q T_3)(s)\| \leqslant 1/\varepsilon, \qquad \text{all } \operatorname{Re} s > 0.$$