273032 Hilbert Spaces II
Homework #1, due 3.11.2010

34 8.1. Let $E$ be a Banach space and let $A, B, C \in \mathscr{L}(E)$. Show that if $B$ is compact then so is $ABC$.

35 8.2. Let $E$ be an infinite-dimensional Hilbert space and let $A, B \in \mathscr{L}(E)$. Which of the following are true?

(a) If $AB$ is compact then either $A$ or $B$ is compact.
(b) If $A^2 = 0$ then $A$ is compact.
(c) If $A^n = I$ for some $n \in \mathbb{N}$ then $A$ is not compact.

36 8.10. The integral operator $K$ on $L^2(0, \infty)$ is defined by

$$(Kf)(x) = \int_0^\infty k(x+\tau) f(\tau)\, d\tau,$$

where $k: (0, \infty) \to \mathbb{C}$ is a continuous function such that

$$M = \int_0^\infty \tau |k(\tau)|^2\, d\tau < \infty.$$

Prove that $K$ is Hilbert–Schmidt, with Hilbert–Schmidt norm $M^{1/2}$.

Hint: Either use Thm 8.8 directly or imitate the proof of that theorem. You do not have to motivate why the change of order of integration is OK.

37 8.11. Let $K$ be a compact Hermitian operator on a Hilbert space $H$ and let the kernel of $K$ be $\{0\}$. Show that there is a sequence $(K_n)$ of bounded linear operators on $H$ such that

$$K_n K x \to x, \qquad K K_n x \to x \qquad \text{as } n \to \infty.$$

Can the $K_n$ be chosen so that $K_n K \to I$ in the operator norm?

38 9.4. Show that, if $\varphi$ is an eigenfunction corresponding to an eigenvalue $\lambda$ of the Sturm–Liouville system

$$(xf')' + \lambda x f = 0, \qquad 0 < x \leqslant R,$$
$$f \text{ bounded}, \ f(R) = 0,$$

then

$$\lambda \int_0^R x |\varphi(x)|^2\, dx = \int_0^R x |\varphi'(x)|^2\, dx.$$

Deduce that all eigenvalues of the system are positive.

39 9.6. Find the eigenvalues and eigenfunctions of the following Sturm–Liouville systems.

(a) $f'' + \lambda f = 0, \qquad f'(0) = 0 = f(1).$
(c) $f'' + f' + \lambda f = 0, \qquad f(0) = 0 = f(1).$



and hence

$$x = (I + GK)^{-1} G u.$$

This shows that the compound plant, consisting of the original plant together with the feedback controller $K$ in the configuration of Diagram 14.2 is

$$(I + GK)^{-1} G.$$

To take an example of extreme simplicity, the function

$$G(s) = \frac{1}{s-3}$$

represents an unstable system, having a pole at $s = 3$. If we choose $K(s)$ to be constant and equal to 4, we find

$$(I + GK)^{-1} G(s) = \frac{1}{s+1},$$

and so the feedback controller $K$ stabilizes the plant.

Returning once again to Diagram 14.2 we observe that the input which actually reaches the plant is

$$\begin{aligned} u - Kx &= u - K(I + GK)^{-1} G u \\ &= u - (I + KG)^{-1} K G u \\ &= (I + KG)^{-1} u. \end{aligned}$$

If it happens that the transfer function $(I + KG)^{-1}$ has a pole in the right half plane then the compound plant will have instabilities inside it, even though its overall transfer function may be free of right half plane poles. Similar considerations lead us to say that the feedback system of Diagram 14.2 is *internally stable* if $(I + GK)^{-1}$, $(I + GK)^{-1} G$, $K(I + GK)^{-1}$ and $(I + KG)^{-1}$ are all analytic in the right half plane (they are allowed to have removable singularities). Suppose the plant $G$ is given to us and we seek a

**Diagram 14.2**