273031 Hilbert Spaces I
Homework #5, 6.10.2010

(25) 7.1. Let $(e_n)_1^\infty$ be a complete orthonormal sequence in a Hilbert space H and let $\lambda_n \in \mathbb{C}$ for $n \in \mathbb{N}$. Show that there is a bounded linear operator D on H such that $De_n = \lambda_n e_n$, all $n \in \mathbb{N}$, if and only if (λ_n) is a bounded sequence. What is $\|D\|$, when defined?

Hint: For each $x = \sum_{n=1}^{N} \langle x, e_n\rangle e_n$ in the linear span of $\{e_n\}$ we must have $Dx = \sum_{n=1}^{N} \lambda_n \langle x, e_n\rangle e_n$

(26) 7.9. An operator is said to have *rank 1* if its range is one-dimensional. Let T be a bounded operator of rank 1 on a Hilbert space H, and let ψ be a non-zero vector in the range of T. Show that there exists $\varphi \in H$ such that

$$Tx = (x, \varphi)\psi, \qquad \text{all } x \in H$$

and that

$$\|T\| = \|\varphi\| \|\psi\|.$$

(27) 7.12. Show that the operator D of Problem 7.1, assumed bounded, is invertible if and only if

$$\inf_n |\lambda_n| > 0.$$

What is $\|D^{-1}\|$, when applicable?

Hint: Try to find an explicit formula for D^{-1}, by first considering vectors in the linear span of $\{e_n\}$.

(28) 7.14. Show that the rank 1 operator T of Problem 7.9 satisfies $T^2 = (\psi, \varphi)T$. Hence show that, if $(\psi, \varphi) \neq 1$, $I - T$ is invertible and find $(I-T)^{-1}$.

(29) 7.21. Let H, K be Hilbert spaces and let $T \in \mathscr{L}(H, K)$. Show that T^*T is Hermitian and that

$$\|T^*T\| = \|T\|^2$$

(use Theorem 7.18).

(30) 7.29. Let H be a Hilbert space and let $A \in \mathscr{L}(H)$. Prove that $(\text{Range } A)^\perp = \text{Ker } A^*$ and that $(\text{Ker } A)^\perp$ is the closure of Range A^*. Prove also that $\text{Ker } A^*A = \text{Ker } A$.

Hint: $\overline{V} = (V^\perp)^\perp$ for every subspace V.

differential equation, though typically several different linear models will be needed for a single plant: clearly different linear approximations will be appropriate to an aeroplane in steady flight and during landing.

Let us consider a plant modelled by the equations

$$Lx = Mu,$$

where L, M are linear differential operators, x and u being functions of time with values in Euclidean spaces X and U. It is natural to assume that the physical characteristics of the plant do not change over the period during which control is to be exercised, and this corresponds to assuming that L, M are linear differential operators with constant coefficients. On taking Laplace transforms we obtain (assuming, for simplicity, zero initial conditions)

$$A(s)\bar{x}(s) = B(s)\bar{u}(s) \tag{14.1}$$

where $\bar{x}(s), \bar{u}(s)$ are the Laplace transforms of $x(t), u(t)$ and $A(s), B(s)$ are matrix-valued polynomials in s. An adequate model must contain enough information to determine $x(\cdot)$ from $u(\cdot)$, so we may take it that (14.1) may be solved to give

$$\bar{x}(s) = G(s)\bar{u}(s), \tag{14.2}$$

where $G(s)$ is a matrix of rational functions in s. $G(s)$ is called the *transfer function matrix*, or simply *transfer function*, of the system.

What an encouraging conclusion for the pure mathematician! Such a marvel of engineering as a modern aircraft, with all its complexity and power, can be usefully represented by a few rational matrix functions. True, they may be rather large; a lot of information is doubtless encoded in their coefficients and someone may have to do a lot of work to estimate them, but in principle, with the great structures of linear algebra, complex analysis, ring theory and functional analysis to support us, we should feel at home with them.

We have seen that an undesirable feature of a physical device is instability. We must translate this into a statement about transfer functions. Consider a system whose state at time t can be represented by a single real number $x(t)$, with a single control input $u(t)$, governed by the equations

$$\ddot{x} + \dot{x} - 6x = u(t), \tag{14.3}$$
$$x(0) = \dot{x}(0) = 0.$$

The transfer function of this plant is

$$G(s) = \frac{1}{s^2 + s - 6} = \frac{1}{(s-2)(s+3)},$$