273031 Hilbert Spaces I
Home work #4, 29.9.2010

(19) 6.1. Let $-\infty < a < c < b < \infty$. Show that the linear functional F on $C[a,b]$ defined by

$$F(x) = x(c), \qquad x \in C[a,b],$$

is continuous with respect to the supremum norm, but not with respect to the $L^2(a,b)$ norm (restricted to $C[a,b]$).

Does it make sense to define a functional G on $L^2(a,b)$ by

$$G(x) = x(c)?$$

(20) 6.5. Find the norm of the linear functional

$$F(x) = \int_0^1 tx(t)\,\mathrm{d}t$$

on $(C[0,1], \|\cdot\|_\infty)$. Find also an element of $C[0,1]$ at which F attains its norm: that is, an element x of unit norm such that

$$|F(x)| = \|F\|.$$

(21) 6.6. Let

$$E = \{x \in C[0,1] : x(1) = 0\},$$

and let G be the restriction to E of the linear functional F of the preceding problem. Show that $\|G\| = \|F\|$, but that G does not attain its norm on $(E, \|\cdot\|_\infty)$.

(22) 7.3. Let X, Y be compact Hausdorff spaces and $\alpha: X \to Y$ be a continuous mapping. Let $C(X)$ be the Banach space of continuous $\mathbb{C}$-valued functions on X with supremum norm and let $T: C(Y) \to C(X)$ be the operation of composition with α – that is,

$$(Tf)(x) = f \circ \alpha(x) = f(\alpha(x)),$$

all $f \in C(Y)$, $x \in X$. Show that T is a bounded linear operator and $\|T\| \leqslant 1$.

(23) 7.7. Let $\mathscr{D}$ be as in Example 7.2(iii). Show that the linear operator

$$\frac{\mathrm{d}}{\mathrm{d}x}: \mathscr{D} \to L^2(-\infty, \infty)$$

is unbounded with respect to the L^2-norm on $\mathscr{D}$ but is bounded with respect to the inner product

$$(f,g) = \int_{-\infty}^{\infty} f(t)\overline{g(t)} + f'(t)\overline{g'(t)}\,\mathrm{d}t$$

on $\mathscr{D}$.

(24) Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space H. For each $x \in H$, define

$$Px = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n.$$

Show that P is an orthogonal projection whose range is the closed linear span of $\{e_1, e_2, \dots\}$. (Thus, the null space of P is the orthogonal complement to $\{e_1, e_2, \dots\}$.)