273031 Hilbert Spaces I
Homework #3, due 22.3.210

(13) 4.9. Find

$$\min_{a,b,c\in\mathbb{C}} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2\, dx$$

(use the solution of Problem 4.7).

(14) 4.10. Let H be a Hilbert space and let M be a closed linear subspace of H, so that M is a Hilbert space with respect to the restriction of the inner product of H. Let $(e_n)_1^\infty$ be a complete orthonormal sequence in M. Show that, for any $x \in H$, the best approximation to x in M is

$$y = \sum_{n=1}^{\infty} (x, e_n) e_n;$$

that is, y satisfies

$$\|x - y\| = \inf_{m\in M} \|x - m\|.$$

(15) 4.13. Let $f \in RL^2$ have Laurent expansion

$$f(z) \sim \sum_{n=-\infty}^{\infty} a_n z^n$$

valid in an annulus containing the unit circle. Let f_+ be the best approximation to f in RH^2. Show that

$$f_+(z) = \sum_{n=0}^{\infty} a_n z^n$$

with respect to the norm of RH^2 (use Problem 4.10).

(16) 5.1. Find the Fourier series of the following functions on $[-\pi, \pi]$ and use Corollary 5.7 to deduce the stated formulae.

(c) $f(x) = e^{sx}$; $\qquad \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{n^2+s^2} = \frac{\pi}{s}\coth \pi s.$

(17) 5.5. Show that if $f \in L^2(-\pi, \pi)$ has Fourier series

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2\, dx = |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty} |a_n|^2 + |b_n|^2.$$

(18) 5.8. Let $-\infty < a < b < \infty$. Show that there is an orthonormal sequence $(f_n)_0^\infty$ in $L^2(a, b)$ such that f_n is a polynomial of degree n. Show further that $(f_n)_0^\infty$ is complete in $L^2(a, b)$ and that each f_n is uniquely determined up to multiplication by a scalar of unit modulus. Take $a = -\pi$, $b = \pi$.

Hint 1: Homework (11)

Hint 2: Every trigonometric function can be approximated by a sequence of polynomials (use Taylor series).

theory, bridging engineering and mathematics. There are a dozen or so international journals on the topic, printing thousands of papers a year. There are papers on general principles common to many cases, others on special features of, say, automatic pilots or the control of chemical processes. People make use of algebra, differential geometry, Riemann surfaces, stochastic processes, non-linear dynamics – every major branch of contemporary mathematics. For illustration consult a recent volume of the *IEEE Transactions on Automatic Control*. Operator theory is but one of the mathematical strands in this tapestry, but lately it has come to play a substantial role in a promising new approach to one of the central concerns of control: how to achieve stabilization and acceptable performance in the face of imperfect knowledge of the system to be controlled. In this chapter I hope to convey an inkling of how this connection is made by means of a drastically simplified presentation.

The first step must be to find a mathematical representation of the physical system which it is desired to control, be it a ship, a steam engine, a chemical works or whatever. This stage takes up a high proportion of the time and effort of the working control engineer, but here we shall assume it has been done. The jargon word, which does for both the physical system and its mathematical representation, is the *plant*. We suppose that the state of the plant at any instant can be described by a finite sequence of real numbers. For example, in the case of an automatic pilot we might take the parameters of interest to be the co-ordinates of the centre of gravity and the Euler angles of the aircraft with respect to suitable co-ordinate axes, together with the derivatives of these quantities with respect to time. The state of the system would then be represented by a vector in $\mathbb{R}^{12}$. We suppose further that we can influence the behaviour of the system by means of a finite number of 'control inputs', each represented by a real number. Thus, in the case of an aircraft, we can adjust the thrust of the engines and the settings of the rudder, elevons, etc. The state of the system and the control inputs at time t are consequently represented by vectors $x(t) \in X$, $u(t) \in U$ where X and U are Euclidean spaces of suitable dimension. A model of the control system consists of a relation between $x(\cdot)$ and $u(\cdot)$, most commonly a differential equation. In the flight example there are six basic equations, corresponding to components of force along and torque about each of the three co-ordinate axes. These equations involve trigonometric functions of the Euler angles, and so are non-linear. However, for the analysis of small deviations from an operating equilibrium we may replace the non-linear terms by linear approximations, just as one replaces $\sin\theta$ by θ in discussing the small oscillations of a simple pendulum. Control is thus usually based on modelling the plant by a linear