273031 Hilbert Spaces I

Home work #2, due 15.9.2010

7 3.3. Show that the normed space $(C(X), \|\cdot\|_\infty)$ of Exercise 2.2 is a Banach space.

8 3.4. Show that $W[a,b]$ (Problem 1.2) is not a Hilbert space (use the indefinite integrals of the functions in Problem 3.2).

Hint: See lecture notes for solution of problem 3.2.

9 3.9. Let E be the Banach space $\mathbb{R}^2$ with norm

$$\|(x_1, x_2)\| = \max\{|x_1|, |x_2|\}.$$

Show that E does not have the closest point property by finding infinitely many points in the closed convex set

$$A = \{(x_1, x_2): x_1 \geqslant 1\}$$

which are at minimal distance from the origin.

10 4.3. Let $f_\lambda(t) = e^{i\lambda t}$, $\lambda, t \in \mathbb{R}$. Show that $(f_\lambda)_{\lambda \in \mathbb{R}}$ is an orthonormal system in the inner product space TP of Problem 1.3. This system cannot be indexed by $\mathbb{N}$, by a theorem of Cantor, so is not an orthonormal sequence.

11 4.6. The Gram–Schmidt process. Let $x_1, x_2, \ldots$ be a sequence of linearly independent vectors in an inner product space. Define vectors e_n inductively as follows.

$$e_1 = x_1/\|x_1\|;$$

$$f_n = x_n - \sum_{j=1}^{n-1} (x_n, e_j)e_j, \qquad n \geqslant 2;$$

$$e_n = f_n/\|f_n\|, \qquad n \geqslant 2.$$

Show that $(e_n)_1^\infty$ is an orthonormal sequence having the same closed linear span as the x_js.

12 4.7. The first three Legendre polynomials are

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \tfrac{1}{2}(3x^2 - 1).$$

Show that the orthonormal vectors in $L^2(-1, 1)$ obtained by applying the Gram–Schmidt process to $1, x, x^2$ are scalar multiples of these.

mathematics is that it has direct application in engineering: it is one of the theoretical components in one approach to the design of certain engineering devices, notably automatic controllers. Those who like their mathematics self-contained and are not inspired by its relation to the physical world may proceed at once to the next chapter.

Gadgets which control machines automatically are all round us and are becoming ever more significant to our society. Their proliferation is usually reckoned to have started around 1788, with the widespread introduction of James Watt's governor. This was a simple mechanical device for regulating the speed of steam engines. Steel balls are suspended on rods attached to a vertical axle driven by the engine. As the angular velocity of the axle increases, the balls rise and this movement is communicated through levers to a valve which reduces the supply of steam to the engine and so restores the angular velocity to a lower value. Contrary to myth, Watt was not the originator of this idea; nor did he contribute to its theory. His success was evidently due to good production engineering: he tried out numerous ways of regulating engines, and was ultimately able to market a governor which worked reliably.

Although Watt's governor was adequate for the requirements of industrial steam engines, the technique fell short of perfection. Astronomers particularly had very stringent requirements for the accuracy of the clockwork mechanisms which compensated for the rotation of the earth and kept their telescopes aligned with the stars. They found that in practice governors brought about angular velocities which, instead of being constant, oscillated about the equilibrium value. In unfavourable circumstances these oscillations could dominate the motion. These phenomena were first discussed and treated mathematically by the Astronomer Royal, George Airy, in 1840. Referring to the operation of frictional forces, Airy wrote: 'In inadvertence of their effect I constructed (some time since) a machine for uniform motion. It went exceedingly well till inequality began to be perceptible; this inequality, though still of an oscillatory character, increased rapidly, and finally became so great as entirely to destroy the character of the previous motion; and the machine (if I may so express myself) became perfectly wild.'

Such behaviour is now known more prosaically as dynamic instability: it remains something to be avoided above all else in controlling devices. Airy's mathematical treatment was not definitive: the foundations of a proper treatment of stability were laid by Maxwell in 1868 (claims are also made for Vishnegradskii and Stodola). Although Maxwell established important principles, he certainly did not say the last word. Since then the subject of control has developed an enormous body of experience and