27 3032 Hilbert Spaces II

Home work # 5, due 1. 12. 2010

This is the last home work.

52 12.7. Let $A, B \in \mathscr{L}(H)$, let $A \geqslant 0$ and let $AB = BA$. Prove

(a) $p(A)B = Bp(A)$ for every scalar polynomial p;

(b) $A^{1/2}B = BA^{1/2}$;

(c) if also $B \geqslant 0$ then $AB \geqslant 0$.

53 12.10. Let $A, B, C \in \mathscr{L}(H)$ and let A be invertible. Show that

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geqslant 0 \quad \text{in } \mathscr{L}(H \oplus H)$$

if and only if $A \geqslant 0$ and $C - B^*A^{-1}B \geqslant 0$.

Hint: Use Schur complements: Show that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}.$$

54 15.6. Let H, K be Hilbert spaces and let $T \in \mathscr{L}(H, K)$. Let x be a maximizing vector for T. Show that $\|T\|^2$ is an eigenvalue of T^*T and that x is a corresponding eigenvector. (Use the result of Problem 12.6.)

55 15.8. Let H, K be Hilbert spaces and let $T \in \mathscr{L}(H, K)$, $T \neq 0$.

(i) Show that any maximizing vector for T^*T is also a maximizing vector for T.

(ii) Show that if x is a maximizing vector for T then Tx is a maximizing vector for T^*.

(iii) Show that if T is compact then T has a maximizing vector.

(iv) Show that if x is a maximizing vector for T then $x \perp \operatorname{Ker} T$ (use Lemma 4.23).

56 16.1. Let A be an $m \times n$ complex matrix and let the singular value decomposition of A be $A = UDV^*$ (so that U, V are unitary matrices and D is a non-negative diagonal matrix). What are the singular values and corresponding Schmidt pairs of the operator from $\mathbb{C}^n$ to $\mathbb{C}^m$ determined by A?

Hint: Find the eigenvalues and eigenvectors of $A^*A = (UDV^*)^*(UDV^*) = VD^*U^*UDV^* = VD^*DV^*$ and $AA^* = \cdots$. Here $UU^* = U^*U = I$ and $VV^* = V^*V = I$ (these are unitary).

57 16.2. Let $A, B \in \mathscr{L}(H, K)$, where H, K are Hilbert spaces, and let $A^*A \leqslant B^*B$. Show that $s_k(A) \leqslant s_k(B)$ for any non-negative integer k.