273032 Hilbert Spaces II

Homework #4, due 24.11.2010

Exam on December 17, at 8.45–13 ?

(47) Solve the problem

$$f''(x) = 1,$$
$$f(0)=0, \quad f(1)+f'(1)=0$$

by using

a) Green's function (1 point)

b) An eigenfunction expansion (2 points)

(48) Use separation of variables to solve the following heat flow problem (a bar of length L, both end points held at fixed temperature).

$$\frac{\partial u(x,t)}{\partial t} = k\,\frac{\partial^2 u(x,t)}{\partial x^2} \qquad 0 \le x \le L, \quad t \ge 0$$
$$u(0,t) = 0 = u(L,t), \qquad t \ge 0$$
$$u(x,0) = h(x). \qquad \text{(3 points)}$$

12.1. Show that the integral operator K on $L^2(0,1)$ with kernel function $k(t,s)=\min(t,s)$ is positive (find the eigenvalues of K and invoke the spectral theorem).

12.5. Show that if $A \in \mathscr{L}(H)$ and $A \geqslant 0$ then $A^n \geqslant 0$ for any $n \in \mathbb{N}$ *without* assuming the existence of a square root of A (consider even and odd n separately). (Assume $A=A^*$)

12.6. Let $A \in \mathscr{L}(H)$, let $A \geqslant 0$ and let $(Ax,x)=0$ for some $x \in H$. Show that $Ax=0$.

Again assume $A=A^*$ (or that the space is complex)

Note: 12.5 and 12.6 are false in real Hilbert spaces, without the assumption $A=A^*$ (counterexample: 90° rotation in $\mathbb{R}^2$)

Furthermore, if we can find such a Q, then on substituting it in (14.5) we shall obtain the desired robustly stabilizing controller.

Determining whether there is such a Q (and finding one if there is) is called by engineers the *model matching problem*. For the special case of scalar functions it was solved long ago. The classical formulation was as an interpolation problem. Suppose that

$$T_2(s)\,T_3(s) = \prod_{j=1}^{n} s - s_j,$$

where $s_1, \ldots, s_n$ are distinct points in the right half plane. Let us for the moment denote by H^∞ the space of functions bounded and analytic in the right half plane (this conflicts slightly with the usage in other chapters, but since the half-plane and disc are related by the Cayley transform $z=(s-1)/(s+1)$, there is no essential difference). The class of functions expressible as $T_1 + T_2QT_3$, for some $Q \in H^\infty$, consists precisely of the functions $F \in H^\infty$ satisfying $F(s_j) = T_1(s_j)$, $1 \leqslant j \leqslant n$. Hence the problem reduces to ascertaining whether there exists a function $F \in H^\infty$ meeting a finite number of interpolation conditions and with supremum norm no greater than $1/\varepsilon$. This is known as the Nevanlinna–Pick problem (see Problem 13.14). Numerous solutions of this problem have been put forward, from around 1917 onwards, many of them either implicitly or explicitly operator-theoretic in character. One method is given in Theorems 15.14 and 15.16. In simple cases solutions can be found by hand calculation (see Problems 15.12, 15.13), but usually computers are required.

The classical results of complex analysis are often not quite what the engineers want. In particular, transition from scalar to matrix-valued functions often involves substantial new difficulties, and an operator-theoretic viewpoint seems to be a definite advantage in coping with these. Operator theorists have studied matrix versions of problems analogous to Nevanlinna–Pick since the 1950s, but the mathematical literature in this area is relatively thin, and much of the detailed work has been done only recently by engineers. Several groups have devised and implemented effective numerical algorithms for solving the model matching problem with matrix-valued functions.

There is a danger that this chapter will leave the impression that the problems of control system design are essentially solved, at least as far as the H^∞ approach goes. This is false. The foregoing account deals only with robust stabilization: a realistic design problem would entail many other considerations, which would lead to difficult mathematical problems. Both theoretical and practical aspects of the H^∞ approach are being intensively studied.