273031 Hilbert Spaces I.
Home work #1, Due 8. 9. 2010

(1) **1.4 *Exercise*** Show that the formula

$$(A, B) = \text{trace}(B^*A)$$

defines an inner product on the space $\mathbb{C}^{m\times n}$ of $m \times n$ complex matrices, where $m, n \in \mathbb{N}$ and B^* denotes the conjugate transpose of B. □

(2) 1.2. Let $a < b$ in $\mathbb{R}$. Show that the space $W[a,b]$ of continuously differentiable functions on $[a,b]$, with values in $\mathbb{C}$, is an inner product space with respect to pointwise addition and scalar multiplication, and inner product

$$(f,g)_W = \int_a^b f(t)\overline{g(t)} + f'(t)\overline{g'(t)}\,dt.$$

(3) 1.8. Prove that, for any continuously differentiable function f on $[-\pi, \pi]$,

$$\left|\int_{-\pi}^{\pi} f(t)\cos t - f'(t)\sin t\,dt\right| \leqslant \sqrt{(2\pi)}\left\{\int_{-\pi}^{\pi} |f(t)|^2 + |f'(t)|^2\,dt\right\}^{1/2}.$$

Hint: Cauchy–Schwarz inequality.

(4) 2.4. Prove that there is no inner product on $C[0,1]$ such that $(f,f)^{1/2} = \|f\|_\infty$ for all f (show that the parallelogram law does not hold for $\|\cdot\|_\infty$).

(5) 2.7. c_0 denotes the subspace of ℓ^∞ comprising all sequences (x_n) which tend to zero as $n \to \infty$. Prove that c_0 is closed in ℓ^∞ with respect to $\|\cdot\|_\infty$.

(6) 2.9. Show that c_0 (Problem 2.7) is the closed linear span in ℓ^∞ of the set $\{e_n : n \in \mathbb{N}\}$, where e_n is the sequence with nth term 1 and all other terms zero.

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Interlude: complex analysis and operators in engineering

Let us pause a while from the technicalities of spaces and operators and reflect on the place of functional analysis in the wider picture. On the strength of remarks earlier in the book about the genesis of the subject we should certainly expect that functional analysis would provide a framework in which to formulate, discuss and (sometimes) solve problems in the description of phenomena on the basis of classical physics. The hanging chain example of Chapters 9–11 is intended to illustrate this aspect. But this is far from being the only role of the subject. It often happens that concepts introduced and developed for one purpose subsequently provide the key to understanding a quite different set of problems. Basic Hilbert space theory is at once close enough to our geometric intuition to be readily assimilated and advanced enough to provide powerful tools for deepening our knowledge over a wide area of mathematics. It is routinely used in differential geometry, complex analysis, number theory – indeed, almost every branch. As we learn more mathematics we come to appreciate better the inter-relationship between its parts, but in view of the brevity of life no individual can fully grasp all the ways in which a particular intellectual current flows through the body of mathematics and on into science. We can still gain some understanding of the process by studying particular instances, as we form conceptions of the natural and social orders from the relatively few places, things and people we know. Chapters 12–16 describe an application of functional analysis to complex function theory. The problem it solves is a natural one, of independent interest: that is, it is a problem which would (and did) occur to complex analysts working without the perspective of functional analysis. It is even a fundamental problem, in the sense that it often recurs as a step in attacks on more advanced problems. The feature of this problem which makes it especially suitable as an illustration of the workings of