This example shows clearly the difference between the topological approach to dynamics that we will adopt in the sequel and the measure theoretic approach. In a topological sense, an open, dense subset is considered "large." These sets may or may not be large in a measure theoretic sense, i.e., in the sense of total length.

Exercises

1. Decide whether each of the following functions are one-to-one, onto, homeomorphisms, or diffeomorphisms on their domains of definition.

a. $f(x) = x^{5/3}$

b. $f(x) = x^{4/3}$

c. $f(x) = 3x + 5$

d. $f(x) = e^x$

e. $f(x) = 1/x$

f. $f(x) = 1/x^2$

2. Identify which of the following subsets of $\mathbf{R}$ are closed, open, or neither.

a. $\{x | x \text{ is an integer }\}$

b. $\{x | x \text{ is a rational number }\}$

c. $\{x | x = \frac{1}{n} \text{ for some natural number } n\}$

d. $\{x | \sin(\frac{1}{x}) = 0\}$

e. $\{x | x \sin(\frac{1}{x}) = 0\}$

f. $\{x | \sin(\frac{1}{x}) > 0\}$

3. Prove that the set of rational numbers of the form $p/2^n$ for $p, n \in \mathbf{Z}$ is dense in $\mathbf{R}$.

The goal of the next few exercises is to construct special functions which will be useful later when we perturb or change slightly a given function. These functions are called "bump functions." Define

$$B(x) = \begin{cases} \exp(-1/x^2) & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

4. Sketch the graph of $B(x)$.

5. Prove that $B'(0) = 0$.

This example shows clearly the difference between the topological approach to dynamics that we will adopt in the sequel and the measure theoretic approach. In a topological sense, an open, dense subset is considered "large." These sets may or may not be large in a measure theoretic sense, i.e., in the sense of total length.

Exercises

1. Decide whether each of the following functions are one-to-one, onto, homeomorphisms, or diffeomorphisms on their domains of definition.

a. $f(x) = x^{5/3}$

b. $f(x) = x^{4/3}$

c. $f(x) = 3x + 5$

d. $f(x) = e^x$

e. $f(x) = 1/x$

f. $f(x) = 1/x^2$

2. Identify which of the following subsets of $\mathbf{R}$ are closed, open, or neither.

a. $\{x | x \text{ is an integer }\}$

b. $\{x | x \text{ is a rational number }\}$

c. $\{x | x = \frac{1}{n} \text{ for some natural number } n\}$

d. $\{x | \sin(\frac{1}{x}) = 0\}$

e. $\{x | x \sin(\frac{1}{x}) = 0\}$

f. $\{x | \sin(\frac{1}{x}) > 0\}$

3. Prove that the set of rational numbers of the form $p/2^n$ for $p, n \in \mathbf{Z}$ is dense in $\mathbf{R}$.

The goal of the next few exercises is to construct special functions which will be useful later when we perturb or change slightly a given function. These functions are called "bump functions." Define

$$B(x) = \begin{cases} \exp(-1/x^2) & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$$

4. Sketch the graph of $B(x)$.

5. Prove that $B'(0) = 0$.

6. Inductively prove that $B^{(n)}(0) = 0$ for all n. Conclude that $B(x)$ is a C^∞ function.

7. Modify $B(x)$ to construct a C^∞ function $C(x)$ which satisfies

a. $C(x) = 0$ if $x \le 0$.

b. $C(x) = 1$ if $x \ge 1$.

c. $C'(x) > 0$ if $0 < x < 1$.

8. Modify $C(x)$ to construct a C^∞ *bump function* $D(x)$ on the interval $[a, b]$, i.e., $D(x)$ satisfies

a. $D(x) = 1$ for $a \le x \le b$.

b. $D(x) = 0$ for $x < \alpha$ and $x > \beta$ where $\alpha < a$ and $\beta > b$.

c. $D'(x) \ne 0$ on the intervals (α, a) and (b, β).

9. Use a bump function to construct a diffeomorphism $f: [a, b] \to [c, d]$ which satisfies $f'(a) = f'(b) = 1$ and $f(a) = c$, $f(b) = d$.

§1.3 ELEMENTARY DEFINITIONS

The basic goal of the theory of dynamical systems is to understand the eventual or asymptotic behavior of an iterative process. If this process is a differential equation whose independent variable is time, then the theory attempts to predict the ultimate behavior of solutions of the equation in either the distant future ($t \to \infty$) or the distant past ($t \to -\infty$). If the process is a discrete process such as the iteration of a function, then the theory hopes to understand the eventual behavior of the points $x, f(x), f^2(x), \ldots, f^n(x)$ as n becomes large. That is, dynamical systems asks the somewhat nonmathematical sounding question: where do points go and what do they do when they get there? In this chapter, we will attempt to answer this question at least partially for one of the simplest classes of dynamical systems, functions of a single real variable. Functions which determine dynamical systems are also called *mappings*, or *maps*, for short. This terminology connotes the geometric process of taking one point to another. As much of the sequel will in fact be geometric, we will use all of these terms synonymously.

Definition 3.1. The forward orbit of x is the set of points $x, f(x), f^2(x), \ldots$ and is denoted by $O^+(x)$. If f is a homeomorphism, we may define the full

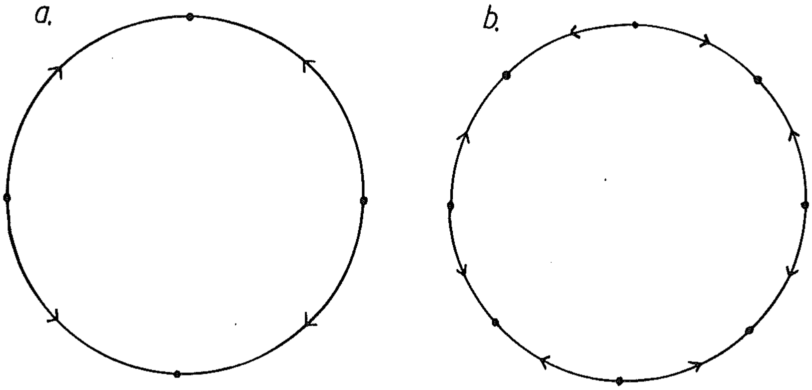

Fig. 3.3. The phase portraits of
a. $f(\theta) = \theta + \epsilon \sin(2\theta)$ and
b. $f(\theta) = \theta + \epsilon \sin(4\theta)$.

Theorem 3.13. *Each orbit T_λ is dense in S^1 if λ is irrational.*

Proof. Let $\theta \in S^1$. The points on the orbit of θ are distinct for if $T_\lambda^n(\theta) = T_\lambda^m(\theta)$ we would have $(n-m)\lambda \in \mathbf{Z}$, so that $n = m$. Any infinite set of points on the circle must have a limit point. Thus, given any $\epsilon > 0$, there must be integers n and m for which $|T_\lambda^n(\theta) - T_\lambda^m(\theta)| < \epsilon$. Let $k = n - m$. Then $|T_\lambda^k(\theta) - \theta| < \epsilon$.

Now T_λ preserves lengths in S^1. Consequently, T_λ^k maps the arc connecting θ to $T_\lambda^k(\theta)$ to the arc connecting $T_\lambda^k(\theta)$ and $T_\lambda^{2k}(\theta)$ which has length less than ϵ. In particular it follows that the points $\theta, T_\lambda^k(\theta), T_\lambda^{2k}(\theta), \ldots$ partition S^1 into arcs of length less than ϵ. Since ϵ was arbitrary, this completes the proof.

q.e.d.

Exercises

1. Use a calculator to iterate each of the following functions (using an arbitrary initial value) and explain these results.

a. $C(x) = \cos(x)$
b. $S(x) = \sin(x)$
c. $E(x) = e^x$
d. $F(x) = \frac{1}{e} e^x$
e. $A(x) = \arctan(x)$

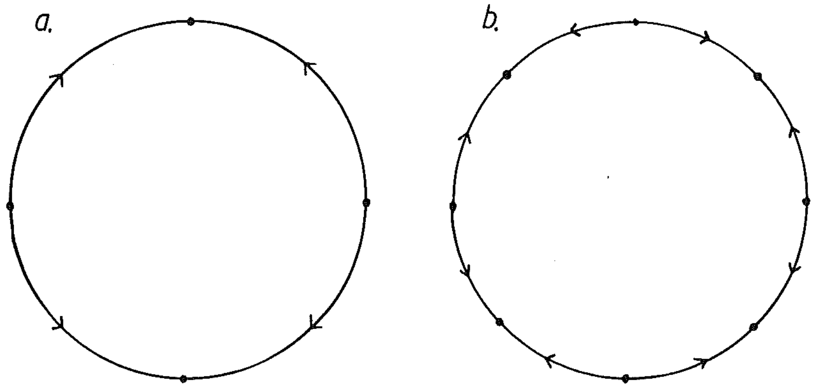

Fig. 3.3. The phase portraits of
a. $f(\theta) = \theta + \epsilon \sin(2\theta)$ and
b. $f(\theta) = \theta + \epsilon \sin(4\theta)$.

Theorem 3.13. *Each orbit T_λ is dense in S^1 if λ is irrational.*

Proof. Let $\theta \in S^1$. The points on the orbit of θ are distinct for if $T_\lambda^n(\theta) = T_\lambda^m(\theta)$ we would have $(n-m)\lambda \in \mathbf{Z}$, so that $n = m$. Any infinite set of points on the circle must have a limit point. Thus, given any $\epsilon > 0$, there must be integers n and m for which $|T_\lambda^n(\theta) - T_\lambda^m(\theta)| < \epsilon$. Let $k = n - m$. Then $|T_\lambda^k(\theta) - \theta| < \epsilon$.

Now T_λ preserves lengths in S^1. Consequently, T_λ^k maps the arc connecting θ to $T_\lambda^k(\theta)$ to the arc connecting $T_\lambda^k(\theta)$ and $T_\lambda^{2k}(\theta)$ which has length less than ϵ. In particular it follows that the points $\theta, T_\lambda^k(\theta), T_\lambda^{2k}(\theta), \ldots$ partition S^1 into arcs of length less than ϵ. Since ϵ was arbitrary, this completes the proof.

q.e.d.

Exercises

1. Use a calculator to iterate each of the following functions (using an arbitrary initial value) and explain these results.

a. $C(x) = \cos(x)$
b. $S(x) = \sin(x)$
c. $E(x) = e^x$
d. $F(x) = \frac{1}{e}e^x$
e. $A(x) = \arctan(x)$

2. Using the graph of the function, identify the fixed points for each of the maps in the previous Exercise.

3. List all periodic points for each of the following maps. Then use the graph of $f(x)$ to sketch the phase portrait of $f(x)$ on the indicated interval.

a. $f(x) = -\frac{1}{2}x, \quad -\infty < x < \infty$
b. $f(x) = -3x, \quad -\infty < x < \infty$
c. $f(x) = x - x^2, \quad 0 \leq x \leq 1$
d. $f(x) = \frac{\pi}{2}\sin x, \quad 0 \leq x \leq \pi$
e. $f(x) = -x^3, \quad -\infty < x < \infty$
f. $f(x) = \frac{1}{2}(x^3 + x), \quad -1 \leq x \leq 1$

4. Identify the stable sets of each of the fixed points for the maps in the previous Exercise.

5. For each of the following functions, list all critical points and decide whether each is degenerate or non-degenerate.

a. $f(x) = x^3 - x$
b. $S(x) = \sin(x)$
c. $f(x) = x^4 - 2x^2$
d. $g(x) = x^3 + x^4$

6. Describe the phase portrait of the map of the circle given by

$$f(\theta) = \theta + \frac{\pi}{n} + \epsilon \sin(n\theta)$$

for $0 < \epsilon < 1/n$.

7. Prove that a homeomorphism of $\mathbf{R}$ can have no periodic points with prime period greater than 2. Give an example of a homeomorphism that has a periodic point of period 2.

8. Prove that a homeomorphism cannot have eventually periodic points.

9. Let $S: S^1 \to S^1$ be given by $S(\theta) = \theta + \omega + \epsilon\sin(\theta)$ where ω and ϵ are constants. Prove that S is a homeomorphism of the circle if $|\epsilon| < 1$.

10. Let $f(\theta) = 2\theta$ be the map of S^1 discussed in Example 3.4. Prove that periodic points of f are dense in S^1.

11. Prove that eventually fixed points for the map in Exercise 10 are also dense in S^1.

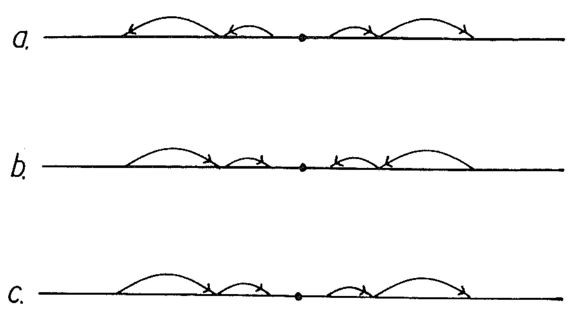

Fig. 4.5. The phase portraits of a. $f(x) = x + x^3$, b. $f(x) = x - x^3$, c. $f(x) = x + x^2$.

has a *weakly* attracting fixed point at 0. In c., the map $f(x) = x + x^2$ is weakly repelling from the right but weakly attracting from the left.

Most maps have only hyperbolic periodic points, as we shall see later. However, non-hyperbolic periodic points often occur in families of maps. When this happens, the periodic point structure often undergoes a *bifurcation*. We will deal with bifurcation theory more extensively later, but for now we give several examples.

Example 4.9. Consider the family of quadratic functions $Q_c(x) = x^2 + c$, where c is a parameter. The graphs of Q_c assume three different positions relative to the diagonal depending upon whether $c > 1/4$, $c = 1/4$, or $c < 1/4$. See Fig. 4.6. Note that Q_c has no fixed points for $c > 1/4$. When $c = 1/4$, Q_c has a unique non-hyperbolic fixed point at $x = 1/2$. And when $c < 1/4$, Q_c has a pair of fixed points, one attracting and one repelling. Thus the phase portrait of Q_c changes as c decreases through $1/4$. This change is an example of a bifurcation.

Example 4.10. Let $F_\mu(x) = \mu x(1 - x)$ with $\mu > 1$. F_μ has two fixed points: one at 0 and the other at $p_\mu = (\mu - 1)/\mu$. Note that $F'_\mu(0) = \mu$ and $F'_\mu(p_\mu) = 2 - \mu$. Hence 0 is a repelling fixed point for $\mu > 1$ and p_μ is attracting for $1 < \mu < 3$. When $\mu = 3$, $F'_\mu(p_\mu) = -1$. We sketch the graphs of F_μ^2 for μ near 3. See Fig. 4.7. Note that 2 new fixed points for F_μ^2 appear as μ increases through 3. These are new periodic points of period 2. Another bifurcation has occurred: this time we have a change in $\mathrm{Per}_2(F_\mu)$.

This quadratic family actually exhibits many of the phenomena that are crucial in the general theory. The next section is devoted entirely to this function.

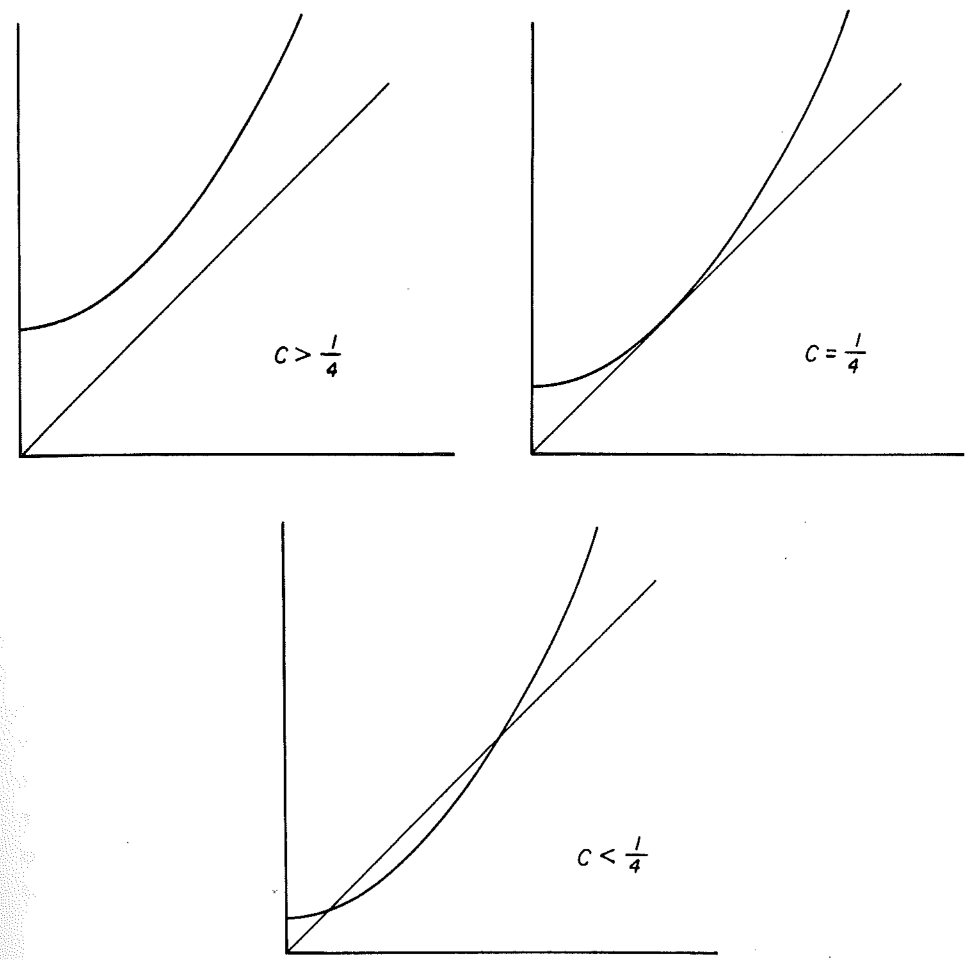

Fig. 4.6. The graphs of $Q_c(x) = x^2 + c$ for $c > 1/4$, $c = 1/4$, and $c < 1/4$.

Exercises

1. Find all periodic points for each of the following maps and classify them as attracting, repelling, or neither. Sketch the phase portraits.

a. $f(x) = x - x^2$

b. $f(x) = 2(x - x^2)$

c. $f(x) = x^3 - \frac{1}{9}x$

d. $f(x) = x^3 - x$

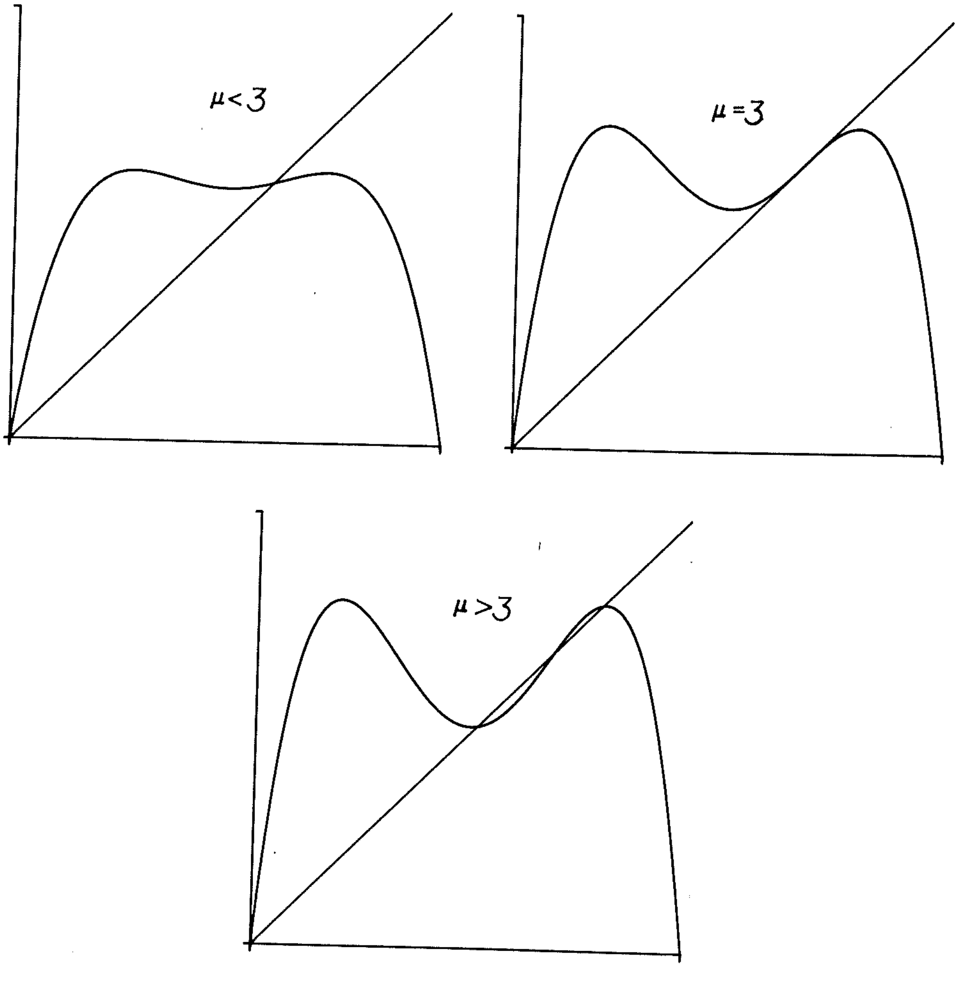

Fig. 4.7. The graphs of $F_\mu^2(x)$ where $F_\mu(x) = \mu x(1-x)$ for $\mu < 3, \mu = 3$, and $\mu > 3$.

e. $S(x) = \frac{1}{2}\sin(x)$

f. $S(x) = \sin(x)$

g. $E(x) = e^{x-1}$

h. $E(x) = e^x$

i. $A(x) = \arctan x$

j. $A(x) = \frac{\pi}{4}\arctan x$

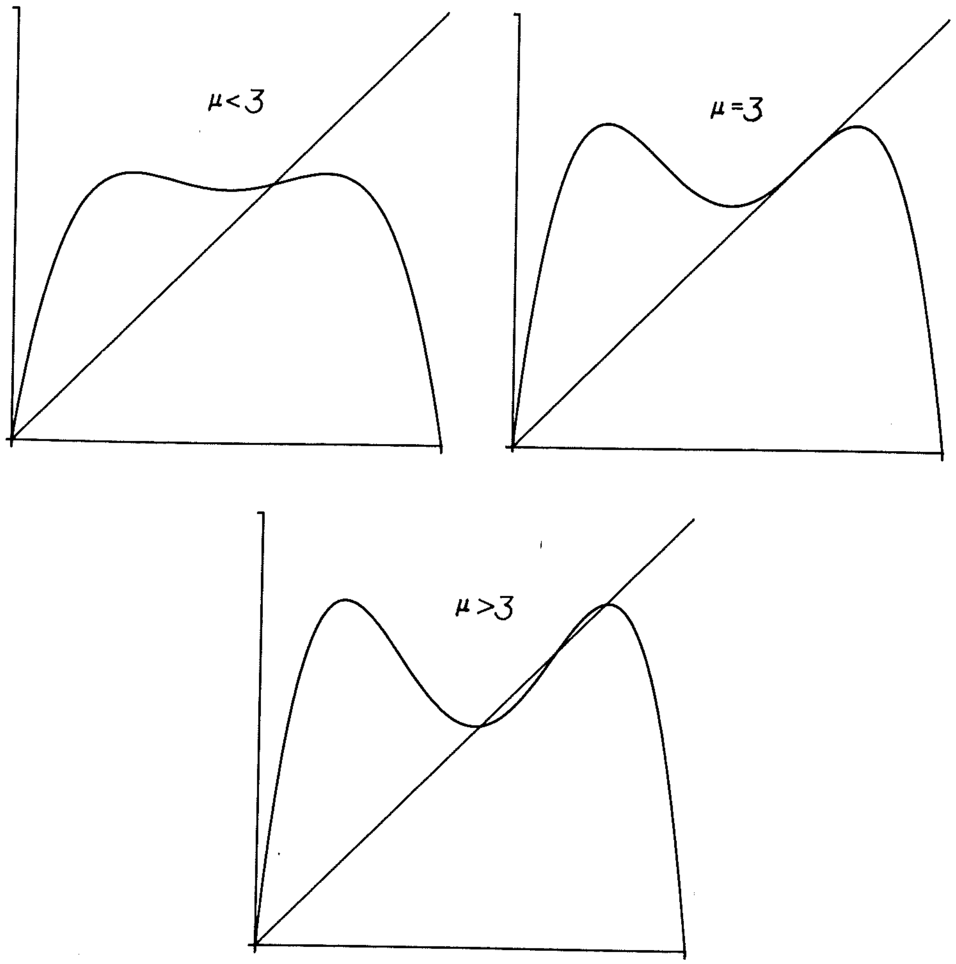

Fig. 4.7. The graphs of $F_\mu^2(x)$ where $F_\mu(x) = \mu x(1-x)$ for $\mu < 3, \mu = 3$, and $\mu > 3$.

e. $S(x) = \frac{1}{2}\sin(x)$

f. $S(x) = \sin(x)$

g. $E(x) = e^{x-1}$

h. $E(x) = e^x$

i. $A(x) = \arctan x$

j. $A(x) = \frac{\pi}{4}\arctan x$

k. $A(x) = -\frac{\pi}{4}\arctan x$

2. Discuss the bifurcations which occur in the following families of maps for the indicated parameter value

a. $S_\lambda(x) = \lambda \sin x, \quad \lambda = 1$

b. $E_\lambda(x) = \lambda e^x, \quad \lambda = 1/e$

c. $E_\lambda(x) = \lambda e^x, \quad \lambda = -e$

d. $Q_c(x) = x^2 + c, \quad c = -3/4$

e. $F_\mu(x) = \mu x(1-x), \quad \mu = 1$

f. $A_\lambda(x) = \lambda \arctan x, \quad \lambda = 1$

g. $A_\lambda(x) = \lambda \arctan x, \quad \lambda = -1$

3. Suppose f is a diffeomorphism. Prove that all hyperbolic periodic points are isolated.

4. Show via an example that hyperbolic periodic points need not be isolated.

5. Find an example of a C^1 diffeomorphism with a non-hyperbolic fixed point which is an accumulation point of other hyperbolic fixed points.

6. Discuss the dynamics of the family $f_\alpha(x) = x^3 - \alpha x$ for $-\infty < \alpha \leq 1$. Find all parameter values where bifurcations occur. Describe how the phase portrait of f_α changes at these points.

7. Consider the linear maps $f_k(x) = kx$. Show that there are four open sets of parameters for which the phase portraits of f_k are similar. The exceptional cases are $k = 0, \pm 1$.

§1.5 AN EXAMPLE: THE QUADRATIC FAMILY

In this section, we will continue the discussion of the quadratic family $F_\mu(x) = \mu x(1-x)$. Actually, we will return to this example repeatedly throughout the remainder of this chapter, since it illustrates many of the most important phenomena that occur in dynamical systems.

there is a sequence of endpoints of the A_k converging to p, or else all points in a deleted neighborhood of p are mapped out of I by some power of F. In the former case, we are done as the endpoints of the A_k map to 0 and hence are in Λ. In the latter, we may assume that F^n maps p to 0 and all other points in a neighborhood of p into the negative real axis. But then F^n has a maximum at p so that $(F^n)'(p) = 0$. By the chain rule, we must have $F'(F^i(p)) = 0$ for some $i < n$. Hence $F^i(p) = 1/2$. But then $F^{i+1}(p) \notin I$ and so $F^n(p) \to -\infty$, contradicting the fact that $F^n(p) = 0$.

Hence we have proved

Theorem 5.6. *If $\mu > 2 + \sqrt{5}$, then Λ is a Cantor set.*

Remark. The theorem is true for $\mu > 4$, but the proof is more delicate.

We have now succeeded in understanding the gross behavior of orbits of F_μ when $\mu > 4$. Either a point tends to $-\infty$ under iteration of F_μ, or else its entire orbit lies in Λ. Hence we understand the orbit of a point under F_μ perfectly well as long as the point does not lie in Λ. In the next section, we will complete the analysis of the dynamics of F_μ by analyzing the dynamics of F_μ on Λ.

When $\mu > 2 + \sqrt{5}$, we have shown that $|F_\mu'(x)| > 1$ on $I_0 \cup I_1$. This implies that $|F_\mu'(x)| > 1$ on Λ. This is a condition similar to the hyperbolicity condition of §3, except that we require $|F_\mu'(x)| \neq 1$ on a whole set, not just at a periodic point. This motivates the definition of a hyperbolic set:

Definition 5.7. A set $\Gamma \subset \mathbf{R}$ is a repelling (resp. attracting) hyperbolic set for f if Γ is closed, bounded and invariant under f and there exists an $N > 0$ such that $|(f^n)'(x)| > 1$ (resp. < 1) for all $n \geq N$ and all $x \in \Gamma$.

The Cantor set Λ for the quadratic map when $\mu > 2 + \sqrt{5}$ is of course a repelling hyperbolic set with $N = 1$.

Exercises

1. Prove that $F_2(x) = 2x(1-x)$ satisfies: if $0 < x < 1$, then $F_2^n(x) \to 1/2$ as $n \to \infty$.

2. Sketch the graph of $F_4^n(x)$ on the unit interval, where $F_4(x) = 4x(1-x)$. Conclude that F_4 has at least 2^n periodic points of period n.

3. Sketch the graph of the tent map

$$T_2(x) = \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 2 - 2x & \frac{1}{2} \leq x \leq 1 \end{cases}$$

there is a sequence of endpoints of the A_k converging to p, or else all points in a deleted neighborhood of p are mapped out of I by some power of F. In the former case, we are done as the endpoints of the A_k map to 0 and hence are in Λ. In the latter, we may assume that F^n maps p to 0 and all other points in a neighborhood of p into the negative real axis. But then F^n has a maximum at p so that $(F^n)'(p) = 0$. By the chain rule, we must have $F'(F^i(p)) = 0$ for some $i < n$. Hence $F^i(p) = 1/2$. But then $F^{i+1}(p) \notin I$ and so $F^n(p) \to -\infty$, contradicting the fact that $F^n(p) = 0$.

Hence we have proved

Theorem 5.6. *If $\mu > 2 + \sqrt{5}$, then Λ is a Cantor set.*

Remark. The theorem is true for $\mu > 4$, but the proof is more delicate.

We have now succeeded in understanding the gross behavior of orbits of F_μ when $\mu > 4$. Either a point tends to $-\infty$ under iteration of F_μ, or else its entire orbit lies in Λ. Hence we understand the orbit of a point under F_μ perfectly well as long as the point does not lie in Λ. In the next section, we will complete the analysis of the dynamics of F_μ by analyzing the dynamics of F_μ on Λ.

When $\mu > 2 + \sqrt{5}$, we have shown that $|F'_\mu(x)| > 1$ on $I_0 \cup I_1$. This implies that $|F'_\mu(x)| > 1$ on Λ. This is a condition similar to the hyperbolicity condition of §3, except that we require $|F'_\mu(x)| \neq 1$ on a whole set, not just at a periodic point. This motivates the definition of a hyperbolic set:

Definition 5.7. A set $\Gamma \subset \mathbf{R}$ is a repelling (resp. attracting) hyperbolic set for f if Γ is closed, bounded and invariant under f and there exists an $N > 0$ such that $|(f^n)'(x)| > 1$ (resp. < 1) for all $n \geq N$ and all $x \in \Gamma$.

The Cantor set Λ for the quadratic map when $\mu > 2 + \sqrt{5}$ is of course a repelling hyperbolic set with $N = 1$.

Exercises

1. Prove that $F_2(x) = 2x(1-x)$ satisfies: if $0 < x < 1$, then $F_2^n(x) \to 1/2$ as $n \to \infty$.

2. Sketch the graph of $F_4^n(x)$ on the unit interval, where $F_4(x) = 4x(1-x)$. Conclude that F_4 has at least 2^n periodic points of period n.

3. Sketch the graph of the tent map

$$T_2(x) = \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 2 - 2x & \frac{1}{2} \leq x \leq 1 \end{cases}$$

on the unit interval. Use the graph of T_2^n to conclude that T_2 has exactly 2^n periodic points of period n.

4. Prove that the set of all periodic points of $T(x)$ are dense in $[0,1]$.

5. Sketch the graph of the baker map

$$B(x) = \begin{cases} 2x & 0 \leq x \leq 1/2 \\ 2x - 1 & 1/2 < x \leq 1 \end{cases}.$$

How many periodic points of period n does B have?

6. The following exercises deal with the family of functions $F(x) = x^3 - \lambda x$ for $\lambda > 0$.
   a. Find all periodic points and classify them when $0 < \lambda < 1$.
   b. Prove that, if $|x|$ is sufficiently large, then $|f^n(x)| \to \infty$.
   c. Prove that if λ is sufficiently large, then the set of points which do not tend to infinity is a Cantor set.

7. Prove that the Cantor Middle-Thirds set described in Example 5.5 is closed, nonempty, perfect, and totally disconnected.

8. Show that, at the n^{th} stage of the construction of the Cantor Middle-Thirds set, the sum of the lengths of the remaining intervals is

$$1 - \frac{1}{3}\left(\sum_{i=0}^{n-1}\left(\frac{2}{3}\right)^i\right).$$

Conclude that the sum of the lengths of these intervals tends to 0 as $n \to \infty$.

9. Construct a Middle-Fifths Cantor set in which the middle fifth of each remaining subinterval of the unit interval is removed. What can be said about the sum of the lengths of the remaining intervals in this case?

10. Let Γ be the Cantor Middle-Thirds set. Prove that the linear map $L(x) = 3x$ maps $\Gamma \cap [0, \frac{1}{3}]$ homeomorphically onto Γ.

11. Generalize Exercise 10 to show that the portion of Γ contained in an interval remaining at the n^{th} stage of the construction of Γ is homeomorphic to Γ.

§1.6 SYMBOLIC DYNAMICS

Our goal in this section is to give a model for the rich dynamical structure of the quadratic map on the Cantor set Λ discussed in the previous section.