Dynamical Systems

A system evolving in time

Mathematical models:

differential equations

difference equations

systems of diff. equations

all kinds of combinations of these

Ex.

$$\frac{dx}{dt} = rx$$

growth models $x(t)$

also: capital in your bank account, interest rate r

$$x(0) = C$$

Discrete-time

$$x(n+1) = r\,x(n)$$

$$x(0) = C$$

Delay-differential equation

$$\frac{dx}{dt} = x(t+h) + r\,x(t) \qquad (h>0)$$

Multivariate

$$\frac{d\vec{X}}{dt} = \vec{F}(\vec{x}, t)$$

$$\frac{dx_1}{dt} = ax_1 + bx_2$$

$$\frac{dx_2}{dt} = -cx_1 + x_2$$

linear 2-dim. system of diff. eq.

Ex.

logistic growth

$$\frac{dP}{dt} = kP(L-P) \qquad k, L \text{ given} > 0$$

$$P(t) = \frac{LP_0 e^{Lkt}}{L - P_0 + P_0 e^{Lkt}}$$

where $P_0 = P(0)$.

Corresponding discrete-time logistic model

$$P(n+1) = k\,P(n)\,(1 - P(n))$$

$$P(0) = x$$

$$P(1) = kx(1-x) \equiv f(x)$$

$$P(2) = f(f(x))$$

$$P(3) = f(f(f(x)))$$

Newton's method

$$Q(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

Problem: To find a root of eq. $Q(x) = 0$.

Newton's recursion scheme

$$x_1 = x_0 - \frac{Q(x_0)}{Q'(x_0)}$$

$$x_2 = x_1 - \frac{Q(x_1)}{Q'(x_1)}$$

$$\vdots$$

works, very well generally, if $Q'(\text{root}) \neq 0$.

$$x_{n+1} = f(x_n) = x_n - \frac{Q(x_n)}{Q'(x_n)}$$

$n = 0, 1, 2, 3, \dots$

Preliminaries from Calculus

$\mathbb{R}, \mathbb{R}^2, \mathbb{R}^n, \mathbb{C}, \mathbb{Q}, \mathbb{Z}$

$I, J, \ldots$ <u>closed</u> intervals : $[a, b] = \{x \in \mathbb{R} \mid a \le x \le b\}$

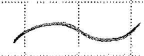
$f : \mathbb{R} \to \mathbb{R}$ function

$f'(x)$ derivative of f at the point x

$f'', f''', \ldots$ higher derivatives

$f \in \mathcal{C}^r$ if $f^{(r)}$, the r^{th} derivative of f, exists and is continuous

$f \in \mathcal{C}^1 \iff f$ <u>smooth</u>

 smooth

 non-smooth (but cont's)

$f \in \mathcal{C}^\infty$ means that <u>all</u> derivatives of f exist

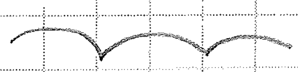
f <u>linear</u> if $f(x) = ax$ d-dim: Ax

<u>affine</u> if $f(x) = ax + b$ d-dim $Ax + b$

<u>piecewise linear</u> if f is affine on a collection of intervals

Ex. $f(x) = |x|$, $f(x) = \begin{cases} 0 & x \le 0 \\ x & x > 0 \end{cases}$

f is piecewise C^2 if it is twice continuously differentiable on a collection of intervals



<u>Def. 2.1</u> f is <u>one-to-one</u> (<u>injective</u>) if $\forall x, y \in \mathbb{R}$

$$x \neq y \Rightarrow f(x) \neq f(y)$$

<u>Def. 2.2</u> $f: I \to J$ (where I, J are intervals).

f is <u>onto</u> (<u>surjective</u>) if $f(I) = J$, i.e.

$\forall y \in J \; \exists x \in I : f(x) = y$.

If f is both 1-to-1 and onto it is <u>bijective</u>.

Then we can define a function $f^{-1}: J \to I$,

the <u>inverse</u> of f, by the rule

$$\forall x \in I, y \in J : \quad f^{-1}(y) = x \iff f(x) = y$$

Ex. $f(x) = x^3$ $\qquad f^{-1}(x) = \sqrt[3]{x}$

$f(x) = x^2, \; x \geq 0$ $\qquad f^{-1}(x) = \sqrt{x}, \; x \geq 0$

<u>Def 2.3</u> Let $f: I \to J$. If f is continuous, one-to-one, onto and f^{-1} is also continuous, then f is a <u>homeomorphism</u>.

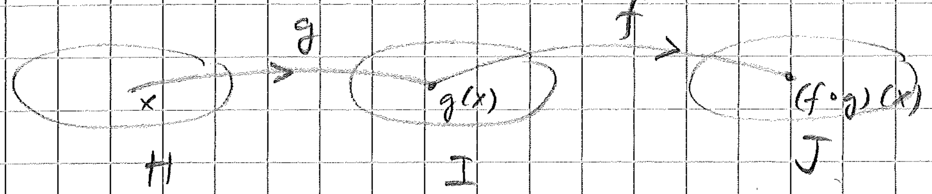
<u>Ex.</u> $f(x) = \tan x$ is a homeomorphism from $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ to $\mathbb{R}$. $\qquad f^{-1}(x) = \arctan x$

Def 2.4 $f: I \to J$. f is a $\mathscr{C}^r$-*diffeomorphism* from I to J if $f \in \mathscr{C}^r$ and also $f^{-1} \in \mathscr{C}^r$.

Composition of functions f, g

$$(f \circ g)(x) = f(g(x))$$

required: $f: I \to J$, $g: H \to I$



Not. $f^2 = f \circ f$, $f^3 = f \circ f \circ f$,

$$f^n = f \circ f^{n-1} = \underbrace{f \circ f \circ \ldots \circ f}_{n \text{ times}}$$

$$f^{-2} = f^{-1} \circ f^{-1} \quad (\text{if exists})$$

$$f^{-n} = \underbrace{f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1}}_{n \text{ times}}$$

Prop 2.5. Chain Rule (Kedjeregeln)

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

If $h(x) = f^n(x)$ then

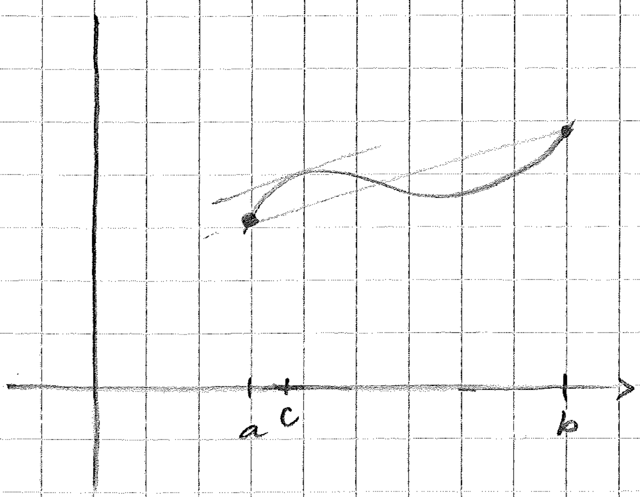
$$h'(x) = f'(f^{n-1}(x)) \cdot f'(f^{n-2}(x)) \cdot \ldots \cdot f'(x)$$

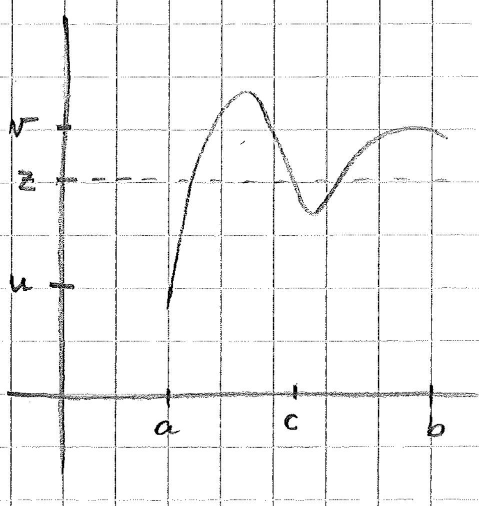
<u>Theorem 2.6</u> (Mean Value Theorem, Medelvärdessatsen)

Suppose $f : [a,b] \to \mathbb{R}$, $f \in \mathscr{C}'$. Then there exists $c \in (a,b)$ such that

$$f(b) - f(a) = f'(c)\,(b-a)$$

<u>Theorem 2.7</u> (Intermediate Value Theorem, Satsen om mellanliggande värden) If $f : [a,b] \to \mathbb{R}$ is continuous and $f(a) = u$, $f(b) = v$, then for any z between u and v there exists $c \in [a,b]$ with $f(c) = z$.





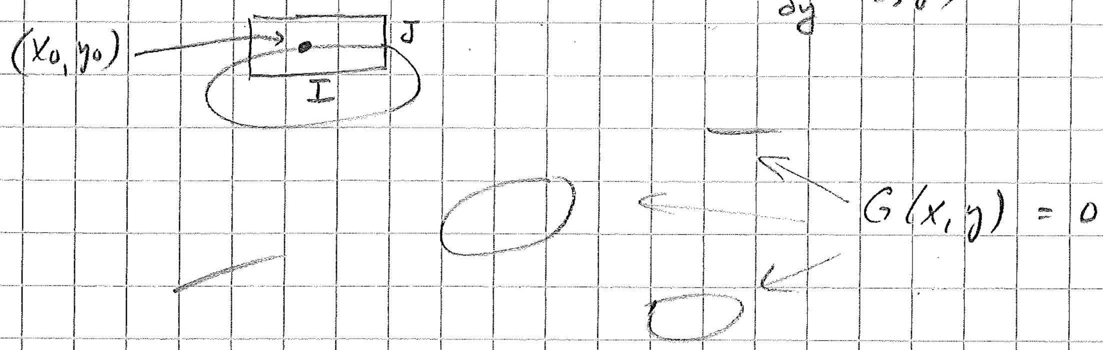
<u>Theorem 2.8</u> (Implicit Function Theorem, Implicita funktioners huvudsats) Suppose $G : \mathbb{R}^2 \to \mathbb{R}^1$ is $\mathscr{C}'$, i.e. both partial derivatives are continuous. Suppose further that

1. $G(x_0, y_0) = 0$

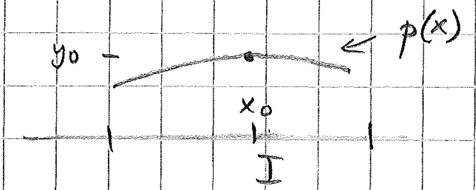
2. $\dfrac{\partial G}{\partial y}(x_0, y_0) \neq 0$

Then there exist open intervals I about x_0 and J about y_0 and a $\mathscr{C}^1$-function $p : I \to J$ such that

1. $p(x_0) = y_0$
2. $G(x, p(x)) = 0$ for $x \in I$.
3. $p'(x_0) = -\dfrac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)}$



Inside $I \times J$



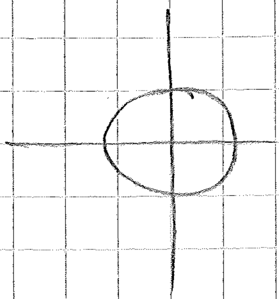
We say that the function $p(x)$ is <u>implicitly defined</u> by $G(x,y) = 0$. The derivative $p'(x)$ is calculated by <u>implicit differentiation</u>

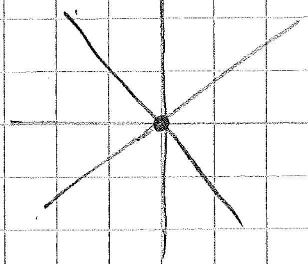
$$\frac{\partial G}{\partial x}(x, p(x)) + \frac{\partial G}{\partial y}(x, p(x))\, p'(x) = 0 .$$

whereby
$$p'(x) = -\frac{\frac{\partial G}{\partial x}(x, p(x))}{\frac{\partial G}{\partial y}(x, p(x))}$$

Ex. $G(x,y) = x^2 + y^2 - 1$

Ex. $G(x,y) = x^3 - xy^2$





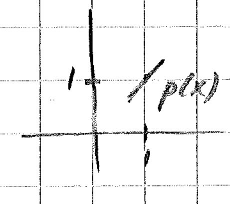
Ex. $G(x,y) = x^5 y^4 - xy^5 - yx^2 + 1$

$G(1,1) = 0$ $\qquad \frac{\partial G}{\partial y}(1,1) = -2 \neq 0$

$\exists\, p(x)$, $\quad p(1) = 1$

$p'(1) = -\frac{2}{-2} = 1$

$G(x, p(x)) = 0$



<u>Prop. 2.11</u> Let $I = [a,b]$ be an interval and let $f : I \to I$ be continuous. Then f has at least one fixed point, i.e. a solution to the eq. $f(x) = x$ in I.

<u>Proof.</u> Let $g(x) = f(x) - x$ (g is cont's.) Then $f(a) \geq a$ and $f(b) \leq b$ so $g(a) \geq 0$ and $g(b) \leq 0$. If either equality is satisfied we have a fixed point there [$g(a) = 0 \iff f(a) = a$ and likewise for b]. If not then $g(a) > 0$ and $g(b) < 0$ and the Intermediate Value Theorem can be invoked to produce a $c \in (a,b)$ with $g(c) = 0$ i.e. $f(c) = c$. ■

Prop 2.12 Let $f: I \to I$ and assume that $|f'(x)| < 1$ for all $x \in I$. Then there is a unique fixed point $c \in I$. Moreover, $|f(x) - f(y)| < |x - y|$, for $x, y \in I$, $x \neq y$.

Proof. The existence of c was proved in Prop. 2.11, the uniqueness remains to be proved. Let d be another fixed point. By the Mean Value Theorem

$$c - d = f(c) - f(d) = f'(\xi)(c - d)$$

for some ξ between c and d. But $|f'(\xi)| < 1$ so $0 < |c - d| < |c - d|$, a contradiction.

The same argument (Mean Value Theorem) shows that

$$|f(x) - f(y)| < |x - y|, \quad x, y \in I, \ x \neq y.$$

Def. 2.13 Let $S \subset \mathbb{R}$. A point $x \in \mathbb{R}$ is a limit point of S if there is a sequence $x_n \in S$ such that $\lim x_n = x$. S is closed if it contains all of its limit points.

Ex. $[a, b]$ is closed

A finite union of closed sets is closed.

Ex. $\bigcup_{n=1}^{\infty} [\frac{1}{n}, 1] = (0, 1]$ is not closed, since 0 is a limit point of the set $(0, 1]$.

Remark. These are topological notions.

Intersections of closed sets are closed. ($\emptyset$ is, by definition, a closed set.)

It can be proved:

If $I_1 \supset I_2 \supset \ldots \supset I_n \supset I_{n+1} \supset \ldots$ are closed, bounded non-empty, then $\bigcap_{n=1}^{\infty} I_n$ is a closed, <u>non-empty</u> set

<u>Def 2.15</u> A set S is <u>open</u> if its complement $\mathbb{R} \setminus S$ is closed. Equivalently, S is open if and only if for any $x \in S$ there is an $\varepsilon > 0$ such that $(x-\varepsilon, x+\varepsilon) \subset S$. (Not proved here.)

A finite intersection of open sets is open. Any union of open sets is open. ($\mathbb{R}$, $\emptyset$ are both open <u>and</u> closed.)

(a, b) is open

Any open set is the union of a countable number of open intervals. (Not proved here.)

If S is a set then $\bar{S}$, the <u>closure</u> of S, is the smallest closed set containing S ($\bar{S} = S \cup \{\text{the limit points of } S\}$)

$\overline{(0,1)} = [0,1]$

If S is closed then $\bar{S} = S$, otherwise $S \subset \bar{S}$.

Def. 2.16 $U \subset S$ is dense in S if $\overline{U} = S$.

The rationals are dense in $\mathbb{R}$.

Ordlista

fixed point	fixpunkt	kiintopiste
closed	sluten	suljettu
closure (of S)	slutna höljet (av S)	sulkeuma
limit point	hopningspunkt	kasautumispiste
open	öppen	avoin
dense	tät	tiheä

Se internetsidan (mest finsk-engelsk)

http://www.math.hut.fi/~kuikkola/matsan/

(engelsk-svensk)

http://www.sm.luth.se/math/molv1.7

http://www.math.uu.se/~vrelis/eng_sv_lexikon.pdf