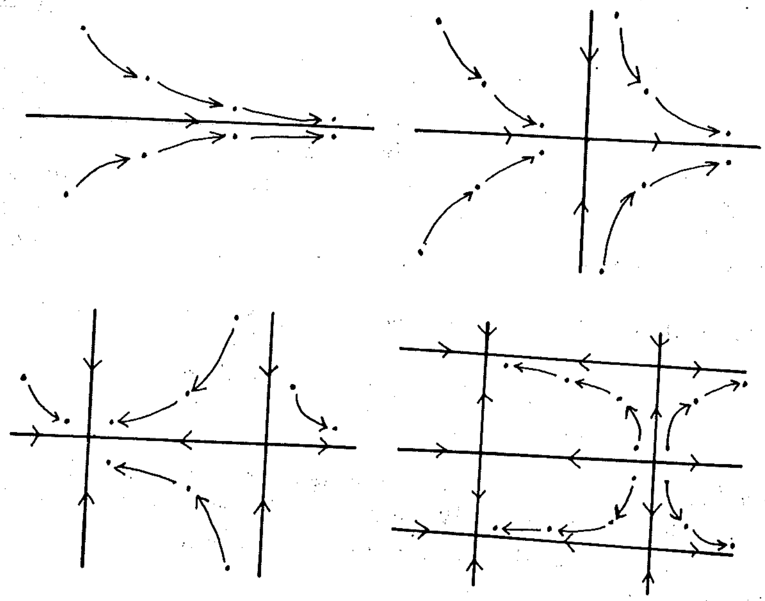

Fig. 8.1. The phase portraits of Q_λ^2.

point and a saddle. Both fixed points undergo period-doubling bifurcations at $\lambda = 2$, so that there are two orbits of period 2 for $\lambda > 2$. As λ passes through 2, the attracting fixed point becomes a saddle, while the saddle becomes a repellor. See Fig. 8.1.

This situation is typical. When $\lambda = 1$, a saddle node bifurcation occurs at the fixed point 0. At this λ-value, $DQ_\lambda(0)$ has an eigenvalue 1 and another eigenvalue less than one in absolute value. When $\lambda = 2$, there are two fixed points for Q_λ, both of which have one eigenvalue -1 and another eigenvalue not equal to one in absolute value.

In higher dimensional systems, there is an additional manner in which a fixed or periodic point may fail to be hyperbolic. When the eigenvalues of the Jacobian matrix are complex but of absolute value one, the fixed point is non-hyperbolic. As long as these eigenvalues are not ± 1, a different type of bifurcation generally occurs.

Analysis of the dynamics of linear maps shows that a bifurcation must occur when an eigenvalue crosses the unit circle. Consider the family of maps

$$L_\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $\lambda > 0$ is the parameter. If $\alpha \neq 0$ and $\lambda < 1$, then 0 is an attracting

Two 1D maps!

$\lambda = 1$: x-coordinate saddle node bifurcation (from no fixed points to two fixed points as $\lambda \searrow$)

$\lambda = 2$: y-coordinate period doubling

$0 < \lambda < 1$ no fixed points, no periodic points

$\lambda = 1$ $(0,0)$ non-hyperbolic fixed point

$1 < \lambda < 2$ $(x_-, 0)$ attr. fixed point

$(x_+, 0)$ saddle

$x_- = x_-^\lambda$
$x_+ = x_+^\lambda$

$\lambda = 2$ $(x_-, 0)$ non-hyperbolic

$(x_+, 0)$ — " —

$\lambda > 2$ (x_-, y_-) attr. 2-periodic point

(x_+, y_+) rep. 2-periodic point

Ex. 8.2. $F_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$

$$F_\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \underbrace{\left(\lambda + \beta(x^2+y^2)\right)}_{g_\lambda(x,y)} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$DF_\lambda(\bar{0}) = \lambda \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Polar coordinates (r, θ)

$$\|F_\lambda(x,y)\| = |\lambda + \beta r^2| \cdot r$$

$$\arg(F_\lambda(x,y)) = \theta + \alpha$$

$$r_1 = \lambda r + \beta r^3$$

$$\theta_1 = \theta + \alpha$$

For $\lambda < 1$ 0 is attracting

$\lambda > 1$ 0 is repelling

$\lambda = 1$ 0 is non-hyperbolic

(regardless whether we regard F_λ or the corresp. map in polar coordinates)

Take $\beta < 0$. Then for $\lambda > 1$ we have an <u>invariant circle</u> at $r = \sqrt{\frac{1-\lambda}{\beta}}$.

Consider one-dim. map

$$r_1 = \lambda r + \beta r^3 \qquad \lambda > 1, \beta < 0$$

Fixed points: 0 repelling

$\sqrt{\frac{1-\lambda}{\beta}}$ attracting

Multiplier: $\lambda + 3\beta r^2 \Big|_{r = \sqrt{\frac{1-\lambda}{\beta}}} = \lambda + 3(1-\lambda)$

$= 3 - 2\lambda.$

 Normal forms

$$x_1 = \alpha x - \beta y + O(2)$$
$$y_1 = \beta x + \alpha y + O(2)$$

$O(2) = O(x^2+y^2)$

Coordinate change : $z = x + iy$
$\bar{z} = x - iy$

(Inverse: $x = \frac{1}{2}(z+\bar{z})$
$y = \frac{1}{2i}(z-\bar{z})$)

Linear term is ~~of~~ of the form $\alpha + i\beta$

$$z_1 = x_1 + iy_1 = \alpha x \overset{+i^2\beta y}{-\beta y} + O(2) + i\beta x + i\alpha y + O(2)$$

$$= (\alpha + i\beta)x + (\alpha + i\beta)iy + O(2)$$
$$= (\alpha + i\beta)z + O(2)$$

Set $\mu = \alpha + i\beta$. We get

$$z_1 = \mu z + O(2)$$
$$\bar{z}_1 = \bar{\mu}\bar{z} + O(2)$$

$O(2)$: terms in $z^2, z\bar{z}, \bar{z}^2, \ldots$ + higher order

 Goal

<u>Theorem 8.4.</u> Suppose $F_\mu(z) = \mu z + O(z)$ where μ is not a k^{th} root of unity for $k = 1, \cdots, 5$. Then there is a neighborhood U of 0 and a diffeomorphism L on U such that the map

$$L^{-1} \circ F_\mu \circ L$$

takes the form

$$z_1 = \mu z + \beta(\mu) z^2 \bar{z} + O(5).$$

Prop. 8.5. Let F_μ be

$$F_\mu(z) = \mu z + \alpha_1 z^2 + \alpha_2 z \bar{z} + \alpha_3 \bar{z}^2 + O(3)$$

where $\mu \neq 0$ and not a root of unity of order k, $k = 1$ or 3. Then $\exists\, U_1$ nbhd of 0, and $L_1 : U_1 \to \mathbb{R}^2$ such that L is a diffeomorphism and

$$L_1^{-1} \circ F_\mu \circ L_1 = G_\mu$$

where

$$G_\mu(z) = \mu z + O(3)$$

Pf. Calculation.

96a $L_1(z) = z + b_1 z^2 + b_2 z\bar{z} + b_3 \bar{z}^2$ sought,

such that

$$F_\mu \circ L_1 = L_1 \circ G_\mu$$

$F_\mu \circ L_1$:

$$\mu z + \alpha_1 z^2 + \alpha_2 z\bar{z} + \alpha_3 \bar{z}^2 + O(3) \Big|_{z = L_1(z)}$$

$$= \mu z + \mu b_1 z^2 + \mu b_2 z\bar{z} + \mu b_3 \bar{z}^2 + \alpha_1 (z + b_1 z^2 + \dots)^2$$
$$+ \alpha_2 (z + b_1 z^2 + \dots)(\bar{z} + \bar{b}_1 \bar{z}^2 + \dots) +$$
$$+ \alpha_3 (\bar{z} + \bar{b}_1 \bar{z}^2 + \dots)^2 + O(3)$$

$$= \mu z + \mu b_1 z^2 + \alpha_1 z^2 + \mu b_2 z\bar{z} + \alpha_2 z\bar{z} +$$
$$+ \mu b_3 \bar{z}^2 + \alpha_3 \bar{z}^2 + O(3)$$

$L_1 \circ G_\mu$:

$$G_\mu(z) + b_1 G_\mu(z)^2 + b_2 G_\mu(z)\overline{G_\mu(z)} + b_3 (\overline{G_\mu z})^2$$
$$= \mu z + O(3) + b_1 \mu^2 z^2 + O(4) + b_2 \mu\bar{\mu} z\bar{z} + O(4)$$
$$+ b_3 \bar{\mu}^2 \bar{z}^2 + O(4)$$

Put: $\mu b_1 + \alpha_1 = b_1 \mu^2 \quad \Rightarrow \quad b_1 = \dfrac{-\alpha_1}{\mu^2 - \mu}$

$\mu\bar{\mu} b_2 = \mu b_2 + \alpha_2 \quad \Rightarrow \quad b_2 = \dfrac{-\alpha_2}{\mu(1-\bar{\mu})}$

$\bar{\mu}^2 b_3 = \mu b_3 + \alpha_3 \quad \Rightarrow \quad b_3 = \dfrac{+\alpha_3}{\bar{\mu}^2 - \mu}$

<u>Prop. 8.6</u> Let G_μ be the map

$$z_1 = \mu z + \beta_1 z^3 + \beta_2 z^2 \bar{z} + \beta_3 z \bar{z}^2 + \beta_4 \bar{z}^3 + O(4),$$

where $\mu \neq 0$ and μ is <u>not</u> a k^{th} root of unity for $k = 2$ or 4. Then $\exists\, U_2$, nbhd of 0, and a diffeomorphism $L_2 : U_2 \to \mathbb{R}^2$ such that

$$L_2^{-1} \circ G_\mu \circ L_2$$

assumes the form H_μ:

$$z_1 = \mu z + \beta_2 z^2 \bar{z} + O(4)$$

<u>Pf</u> L_2 is

$$L_2(z) = z + b_1 z + b_3 z \bar{z}^2 + b_4 \bar{z}^3$$

where

$$b_1 = \frac{-\beta_1}{\mu(1-\mu^2)}$$

$$b_3 = \frac{-\beta_3}{\mu(1-\bar{\mu}^2)}$$

$$b_4 = \frac{-\beta_4}{\mu - \bar{\mu}^3}.$$

<u>Prop. 8.7.</u> Let H_μ be a map of the form

$$H_\mu(z) = \mu z + \beta_2 z^2 \bar{z} + \mathcal{O}(4).$$

Then $\exists\, U_3$, nbhd of 0, and a diffeomorphism $L_3 : U_3 \to \mathbb{R}^2$ such that $L_3^{-1} \circ H_\mu \circ L_3$ is of the form

$$\mu z + \beta_2 z^2 \bar{z} + \mathcal{O}(5)$$

provided μ is not a k^{th} root of unity for $k = 3$ or 5.

Rem. If $\mu^5 = 1$ then the normal form

$$\mu z + \beta |z|^2 z + \gamma \bar{z}^4 + \mathcal{O}(5)$$

may be obtained.

Prop. 8.5, 8.6, 8.7 combine to a proof of Thm 8.4.

In <u>polar coordinates</u> we get

$$\begin{aligned} r_1 &= |\mu| r + \beta(\mu) r^3 + \mathcal{O}(5) \quad \leftarrow \text{invariant circles} \\ \theta_1 &= \theta + \alpha(\mu) + \gamma(\mu) r^2 + \mathcal{O}(5) \end{aligned}$$

where $\mu = |\mu| e^{i\alpha}$.

Eberhard Hopf (1902–1983)
Germany, U.S.

99 **Theorem 8.8.** (Hopf Bifurcation Thm)

Suppose F_λ is a family of maps satisfying

(i) $F_\lambda(0) = 0$ for all λ

(ii) $DF_\lambda(0)$ has eigenvalues $\mu(\lambda), \bar{\mu}(\lambda)$ with $|\mu(0)| = 1$, $\mu(0)$ is <u>not</u> a k^{th} root of unity, $k = 1, 2, 3, 4$ or 5.

(iii) $\dfrac{d}{d\lambda}|\mu(\lambda)| > 0$ at $\lambda = 0$

(iv) In the normal form given by Thm 8.4., the term $\beta(\mu(0)) < 0$.

Then there is an $\varepsilon > 0$ and a closed curve ζ_λ in the form $r = r_\lambda(\theta)$ defined for $0 < \lambda < \varepsilon$ and invariant under F_λ. Moreover, ζ_λ is attracting in a neighborhood of 0 and $\zeta_\lambda \to 0$ as $\lambda \to 0$.

Note. If no $O(5)$ term we have an attracting circle. Proof uses the fact that F is "close to" such a map.