Def 6.6. Let $\bar{p}$ be a hyperbolic fixed point for F and suppose that γ_u is the local unstable manifold at $\bar{p}$. Then the <u>unstable manifold at $\bar{p}$</u>, not.: $W^u(\bar{p})$, is given by

$$W^u(\bar{p}) = \bigcup_{n>0} F^n(\gamma_u)$$

Similarly, if γ_s is the local stable manifold

$$W^s(\bar{p}) = \bigcup_{n>0} F^{-n}(\gamma_s)$$

is the <u>stable manifold at $\bar{p}$</u>.

Ex. 6.7.
$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}x \\ 2y - \frac{15}{8}x^3 \end{pmatrix}$$

$$DF(\bar{0}) = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$$

$$F\begin{pmatrix} 0 \\ t \end{pmatrix} = \begin{pmatrix} 0 \\ 2t \end{pmatrix}$$

$$F\begin{pmatrix} t \\ t^3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t \\ 2t^3 - \frac{15}{8}t^3 \end{pmatrix} = \begin{pmatrix} (\frac{1}{2}t) \\ (\frac{1}{2}t)^3 \end{pmatrix}$$

Thus $W^u(\bar{0}) = \{(x,y) \mid x=0\}$ (y-axis)

$W^s(\bar{0}) = \{(x,y) \mid y = x^3\}$

Ex. 6.8.

Let T be the torus $\left\{(\theta_1, \theta_2) \;\middle|\; \begin{matrix} 0 \le |\theta_i| \le 2\pi \\ i=1,2 \end{matrix}\right\}$

(sides identified)

Let

$$F\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \theta_1 + \epsilon \sin\theta_1 \\ \theta_2 + \epsilon \sin\theta_2 \cos\theta_1 \end{pmatrix}$$

(ε small $\Rightarrow$ F diffeomorphism)

$\theta_1 = \theta_1 + \epsilon \sin\theta_1 \quad \Longrightarrow \quad \theta_1 = 0, \pi$

(2π identified w. 0)

$\theta_2 = \theta_2 + \epsilon \sin\theta_2 \cos\theta_1$

$\Longrightarrow \theta_2 = 0, \pi$

$$DF\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 + \epsilon\cos\theta_1 & 0 \\ -\epsilon\sin\theta_2 \sin\theta_1 & 1 + \epsilon\cos\theta_2 \cos\theta_1 \end{pmatrix}$$

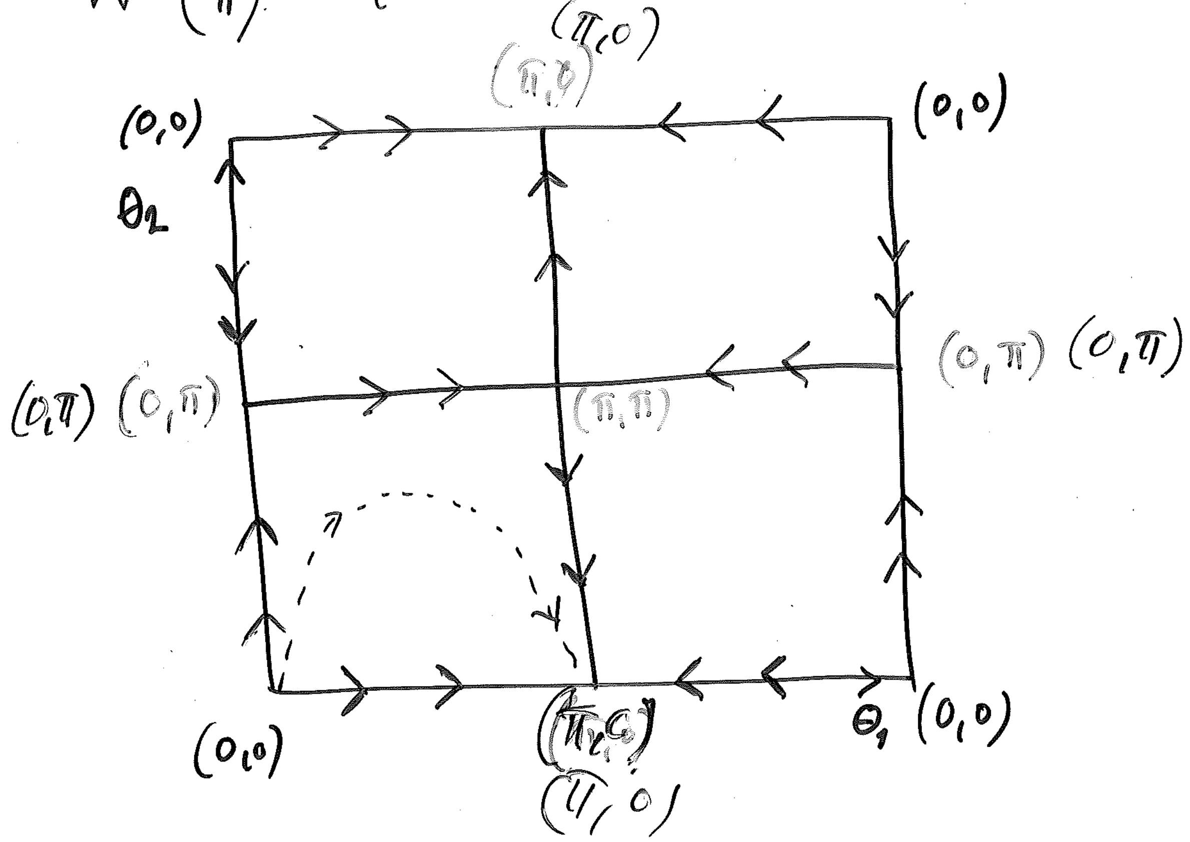
$$DF\begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}1+\epsilon & 0\\ 0 & 1+\epsilon\end{pmatrix}$$ source (for $\varepsilon > 0$ ε small)

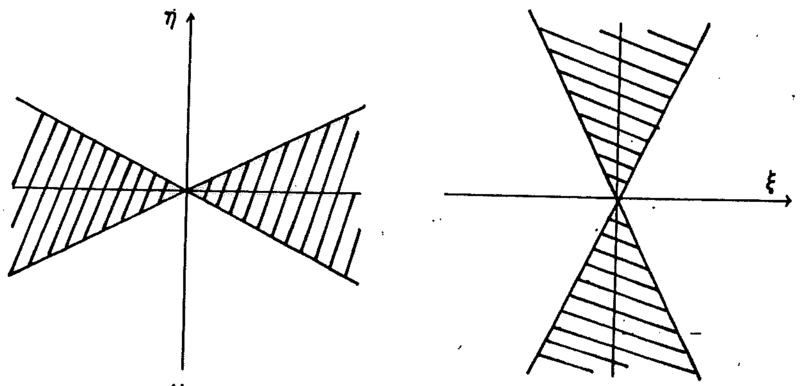
$$DF\begin{pmatrix}0\\\pi\end{pmatrix} = \begin{pmatrix}1+\epsilon & 0\\ 0 & 1-\epsilon\end{pmatrix}$$ saddle

$$DF\begin{pmatrix}\pi\\0\end{pmatrix} = \begin{pmatrix}1-\epsilon & 0\\ 0 & 1-\epsilon\end{pmatrix}$$ sink

$$DF\begin{pmatrix}\pi\\\pi\end{pmatrix} = \begin{pmatrix}1-\epsilon & 0\\ 0 & 1+\epsilon\end{pmatrix}$$ saddle

$$W^u\begin{pmatrix}0\\\pi\end{pmatrix} = \{(\theta_1,\theta_2) \mid \theta_2 = \pi\} = W^s\begin{pmatrix}\pi\\\pi\end{pmatrix}$$





$S^U(q)$ $S^S(q)$

Fig. 6.6. The sector bundles $S^u(q)$ and $S^u(q)$.

See Fig. 6.6.

Note that $DF(0)$ preserves $S^u(0)$ in the sense that if $\mathbf{v} \in S^u(0)$, then $DF(0)\mathbf{v} \in S^u(0)$. Moreover, if

$$\mathbf{v} = \begin{pmatrix} \xi_0 \\ \eta_0 \end{pmatrix}$$

we note that $|\xi_1| = \lambda|\xi_0| > 2|\xi_0|$. Similarly, $(DF(0))^{-1}$ preserves $S^s(0)$ and we have $|\eta_{-1}| = \mu^{-1}|\eta_0| > 2|\eta_0|$.

Since F is at least C^1, the Jacobian matrix $DF(x)$ varies continuously with x and there must therefore be a neighborhood of $\mathbf{0}$ in which the above properties hold. More precisely, there exists $\epsilon > 0$ such that, if $|x|, |y| \le \epsilon$, then

1. $DF(x, y)$ preserves $S^u(x, y)$ and $DF^{-1}(x, y)$ preserves $S^s(x, y)$, i.e., $DF(x, y)\mathbf{v} \in S^u(F(x, y))$ whenever $\mathbf{v} \in S^u(x, y)$.
2. If $(\xi_0, \eta_0) \in S^u(x, y)$, then $|\xi_1| \ge 2|\xi_0|$.
3. If $(\xi_0, \eta_0) \in S^s(x, y)$, then $|\eta_{-1}| \ge 2|\eta_0|$.

The concept of preservation of sector bundles is one that arises whenever hyperbolicity is verified; it is illustrated geometrically in Fig. 6.7.

We will now concentrate on the square B given by $|x|, |y| \le \epsilon$. We say that the curve $\gamma(x) = (x, h(x))$ is a *horizontal curve* in B if

1. h is defined and continuous for $|x| \le \epsilon$
2. $h(0) = 0$

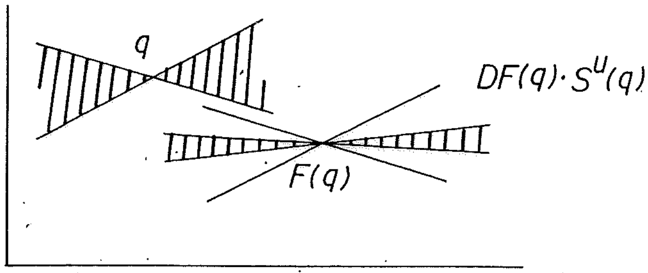


Fig. 6.7. Preservation of the sector bundles.

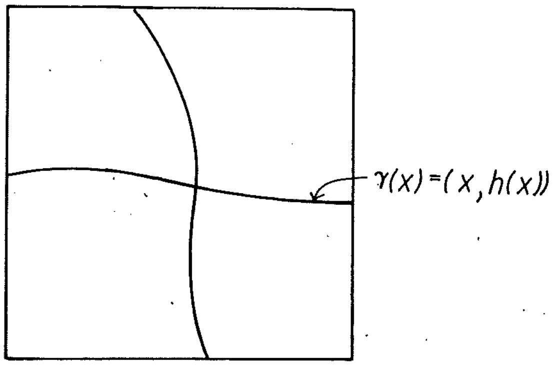


Fig. 6.8. A horizontal and a vertical curve in B.

3. for any x_1, x_2 with $|x_i| \leq \epsilon$, $|h(x_1) - h(x_2)| \leq \frac{1}{2}|x_1 - x_2|$. Note that $\gamma(x)$ is the graph of $h(x)$ which lies in B and is depicted in Fig. 6.8. Reversing the roles of x and y yields a definition of a vertical curve.

Lemma 6.10. *If $\gamma(x) = (x, h(x))$ is a horizontal curve, then the image $F(\gamma(x))$ meets B in a horizontal curve.*

Proof. We first observe that if $(x_1, y_1) = F(\epsilon, h(\epsilon))$, then $x_1 \geq 2\epsilon$. This follows immediately from the fact that $|\xi_1| > 2|\xi_0|$. Similarly, if $(x_1, y_1) = F(-\epsilon, h(-\epsilon))$, then $x_1 < -2\epsilon$. Clearly, $F(\mathbf{0}) = \mathbf{0}$ so that the image curve passes through the origin. Finally, suppose that (x_0, y_0) and (x_0', y_0') lie on $F(x, h(x))$ and $|y_0' - y_0| > \frac{1}{2}|x_0' - x_0|$. Choose α_1, α_2 such that $F(\alpha_1, h(\alpha_1)) = (x_0, y_0)$ and $F(\alpha_2, h(\alpha_2)) = (x_0', y_0')$. Consider the straight line segment ℓ connecting $(\alpha_1, h(\alpha_1))$ to $(\alpha_2, h(\alpha_2))$. The tangent vector to ℓ lies in S^u at each point along ℓ. Now F maps ℓ to a smooth curve connecting (x_0, y_0) to (x_0', y_0'). By the Mean Value Theorem, there is a point on this curve where

Ex. 6.9.

$$G\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\theta_1 - \epsilon \sin\theta_1\\ \theta_2 + \epsilon \sin\theta_2\end{pmatrix}$$

Fixed pts $\begin{pmatrix}0\\0\end{pmatrix}$, $\begin{pmatrix}0\\ \pi\end{pmatrix}$, $\begin{pmatrix}\pi\\0\end{pmatrix}$, $\begin{pmatrix}\pi\\ \pi\end{pmatrix}$

saddle sink source saddle

Remark. Stable and unstable manifolds are preserved under topological conjugacy. (DF is not. Hence hyperbolicity is not.)

<u>Proof sketch</u>

1. Move $\bar{p}$ to $\bar{0}$.
2. Assume $DF(\bar{0}) = \begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$

   with $\lambda > 2$, $0 < \mu < \frac{1}{2}$. (If not, replace F by F^m whose $DF^m = \begin{pmatrix}\lambda^m & 0\\ 0 & \mu^m\end{pmatrix}$.)
3. Define sector bundles at $\bar{q} = (q_1, q_2)$

   $$S^u(\bar{q}), \quad S^s(\bar{q})$$

   $$S^u(\bar{q}) = \{(x,y) \mid |y-q_2| \le \tfrac{1}{2}|x-q_1|\}$$

4. $DF(\bar{0})^{-1}$ preserves $S^s(\bar{0})$
$(DF(\bar{0}))^{4}$ $S^u(\bar{0})$

5. By continuity, $\exists \varepsilon > 0$ so that
$|x|, |y| < \varepsilon \Longrightarrow$
$DF(x,y)^{-1}$ preserves $S^s(x,y)$
$(DF(x,y))$ $S^u(x,y)$

and
$S^s(x,y)$ is mapped inside $S^s(F(x,y))$
$S^u(x,y)$ —— " —— $S^u(F^{-1}(x,y))$

6. $(x, h(x))$ is a <u>horisontal curve</u> in
$B = [-\varepsilon, \varepsilon] \times [-\varepsilon, \varepsilon]$
if $h(0) = 0$ and $|h(x_1) - h(x_2)| \leq$
$\frac{1}{2}|x_1 - x_2|$.

Lemma 6.10. $\gamma = \{(x, h(x)) \mid -\varepsilon \leq x \leq \varepsilon\}$ is a horisontal curve $\Rightarrow$ $F(\gamma(x)) \cap B$ is one, too.

The graph transform
$\gamma(x) \rightsquigarrow F(\gamma(x)) \cap B$.

is a contraction (sup-norm used) in the space of horizontal curves. (Lemma 6.11)

J_u is the unique fixed point of that transformation.

Exists because the space of horizontal curves is a complete metric space!

Finally, uniqueness allows us to go from F^m to F (and take general $\lambda > 1$ and $0 < \mu < 1$).

2.8. The Hopf Bifurcation

Ex. $$L_\lambda \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For $0 < \lambda < 1$ spiral inward if $\alpha \neq 0$

$\lambda = 1$ pure rotation (Jacobi!)

$\lambda > 1$ spiral outward

Ex. 8.1. $$Q_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$$

$$x_1 = e^x - \lambda$$

$$y_1 = -\frac{\lambda}{2} \arctan y$$