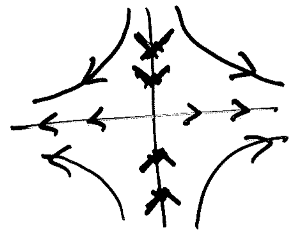
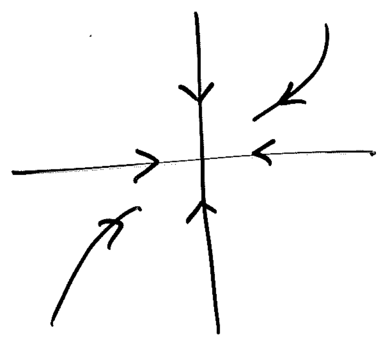
Ex. 2.1. $\begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \bar{x}$



$\bar{0}$ saddle point

Ex. 2.2. $\begin{pmatrix} 2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \bar{x}$

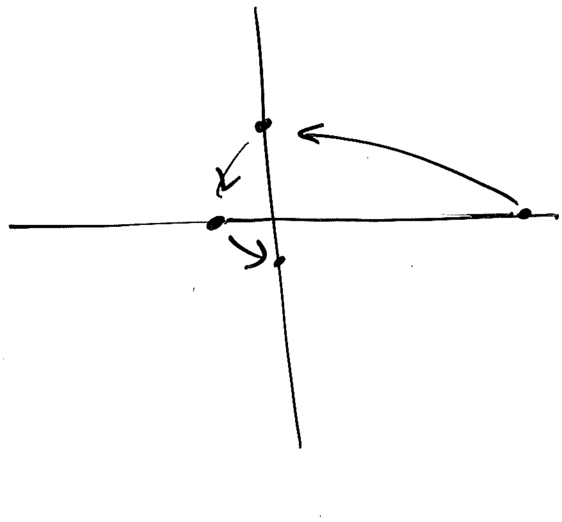
Ex. 2.3. $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \bar{x}$



$\bar{0}$ attractor

Ex 2.4. $\begin{pmatrix} 0 & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \bar{x}$

"inward spiral"



attracting

Ex. 2.5.
$$\begin{pmatrix} 0 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \underline{\text{saddle}}$$

<u>Def. 2.6</u> An invertible linear map is hyperbolic if none of its eigenvalues have absolute value one.

N.B. $A$ invertible $\Leftrightarrow$ eigenvalues $\neq 0$

<u>Prop. 2.7.</u> Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ has all eigenvalues less than one in abs. value. Then
$$L^n(\bar{x}) \to 0, \quad n \to \infty$$
for all $\bar{x} \in \mathbb{R}^3$.

<u>Pf.</u> We need to check each one of the standard forms
$$\begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \eta \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix},$$
$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \quad \text{where } |\lambda|, |\mu|, |\eta| < 1, \quad \alpha^2 + \beta^2 < 1$$

Recall that the 1's above the diagonal may be replaced by any $\varepsilon \neq 0$!

Let $V(x, y, z) = x^2 + y^2 + z^2$.
There exists a $\nu < 1$ so that

$$V \circ L(\bar{x}) \leq \nu V(\bar{x})$$

if $\varepsilon > 0$ is chosen small enough.

Computation in case 1.

$$\alpha^2 x^2 + \beta^2 y^2 - 2\alpha\beta xy + \beta^2 x^2 + \alpha^2 y^2 + 2\alpha\beta xy$$
$$+ \lambda^2 z^2 = \underbrace{(\alpha^2+\beta^2)}_{<1} x^2 + \underbrace{(\alpha^2+\beta^2)}_{<1} y^2 + \underbrace{\lambda^2}_{<1} z^2$$

$$\nu^2 = \max(\alpha^2+\beta^2, \lambda^2)$$

Case 2.

$$\lambda^2 x^2 + \mu^2 y^2 + \eta^2 z^2 \qquad \nu^2 = \max(\lambda^2, \mu^2, \eta^2)$$

Case 3.

[skipped].

Case 4 with $\varepsilon$ small

$$L(\bar{x}) = \begin{pmatrix} \lambda & \varepsilon & 0 \\ 0 & \lambda & \varepsilon \\ 0 & 0 & \lambda \end{pmatrix} \bar{x}$$

$$\lambda^2 x^2 + \varepsilon^2 y^2 + 2\lambda\varepsilon xy + \lambda^2 y^2 + \varepsilon^2 z^2 + 2\lambda\varepsilon yz$$
$$+ \lambda^2 z^2 = \lambda^2(x^2+y^2+z^2) + \varepsilon^2(y^2+z^2) +$$
$$2\lambda\varepsilon(xy+yz) \leq (\lambda^2+\varepsilon^2)(x^2+y^2+z^2)$$
$$+ 2|\lambda\varepsilon|(|xy|+|yz|+|xz|) \qquad (*)$$

$|xy| \le x^2+y^2, \; |yz| \le y^2+z^2, \; |xz| \le x^2+z^2$

$$(*) \le \underbrace{\left(\lambda^2+\varepsilon^2+4|\lambda\varepsilon|\right)}_{\text{may be made } = \nu < 1 \text{ for } \varepsilon \text{ small enough}} \cdot V(\bar{x})$$

$$\therefore \quad V\left(L(\bar{x})\right) \le \nu V(\bar{x})$$

Hence

$$V\left(L^2(\bar{x})\right) \le \nu V\left(L(\bar{x})\right) \le \nu^2 V(\bar{x})$$

and

$$V\left(L^n(\bar{x})\right) \le \nu^n V(\bar{x})$$

Thus

$$V\left(L^n(\bar{x})\right) \to 0 \quad \text{as } n \to \infty$$

$V(\bar{x})$ being $|\bar{x}|^2$ this means $L^n(\bar{x}) \to 0$, too.

<u>Def 2.8</u> $F: \mathbb{R}^n \to \mathbb{R}^n$, a diffeomorphism.

$V: \mathbb{R}^n \to \mathbb{R}$ is a <u>Liapounov fct</u> for $F$ centered at $\bar{p}$ if

1. $V(\bar{x}) > 0$ for $\bar{x} \neq \bar{p}$
2. $V(\bar{p}) = 0$
3. $V \circ F(\bar{x}) \le V(\bar{x})$ with $=$ iff $\bar{x} = \bar{p}$.

Aleksandr Mihailovich (A.M.) Lyapunov *
(1857 – 1918) St. Petersburg, Kharkov

Doctoral thesis: The general problem of the stability of motion, orig. Russian 1892, French transl. 1907, English 1992 (!)

Lyapunov = Liapounov = Liapounoff = ...

<u>Cor. 2.9.</u> Suppose $L: \mathbb{R}^3 \to \mathbb{R}^3$ is linear and all the eigenvalues of $L$ are $> 1$ in abs. value. Then

$$L^{-n}(\bar{x}) \to 0, \quad n \to +\infty.$$

<u>Prop. 2.10</u> Suppose $L$ has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with

1. $|\lambda_1|, |\lambda_2| < 1$
2. $|\lambda_3| > 1$

Then there is a $\underbrace{\text{plane } W^s}_{\text{linear subspace of dim. 2}}$ and a $\underbrace{\text{line } W^u}_{\text{lin. subspace of dim. 1}}$

such that

1. If $\bar{x} \in W^s$ then $L^n(\bar{x}) \to 0$ as $n \to \infty$
2. If $\bar{x} \in W^u$ then $L^n(\bar{x}) \to 0$ as $n \to -\infty$
3. If $\bar{x} \notin W^s \cup W^u$ then $|L^n(\bar{x})| \to \infty$ as $n \to \pm\infty$.

The standard form is either

$$\begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad \begin{array}{l} \alpha^2+\beta^2 < 1 \\ |\lambda_3| > 1 \end{array}$$

or

$$\begin{pmatrix} \lambda_1 & * & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \qquad * \neq 0 \text{ only if } \lambda_1 = \lambda_2$$

The x-y-plane $W^s$ is invariant and the z-axis $W^u$ also.

Apply Cor 2.9. to these subspaces.

Remarks: If two eigenvalues $> 1$ in abs. value, we can make obvious modifications.

In higher dim. the results are similar. Use Jordan normal forms.

<u>Def 2.11</u> $W^s$ is the <u>stable subspace</u> of $L$

$W^u$ is the <u>unstable</u> — " —

## 2.6. The stable and unstable manifold theorem

<u>Def. 6.1</u> A fixed point $p$ for $F: \mathbb{R}^n \to \mathbb{R}^n$ is called <u>hyperbolic</u> if $DF(p)$ has no eigenvalue with abs. value 1.
If $p$ is periodic with period $n$ then $p$ is hyperbolic if $DF^n(p)$ has no eigenvalue with abs. value 1.

Remark. If $p$ is periodic then the eigenvalues of $DF^n$ are the same in each point of the orbit. Use the Chain Rule!
(F diffeomorphism assumed.)

<u>Def. 6.2.</u> Let $F^n(p) = p$.

1. $p$ is a <u>sink</u> or an <u>attracting</u> periodic point if all of the eigenvalues of $DF^n(p)$ are less than one in abs. value.
2. $p$ is a <u>source</u> or <u>repelling</u> periodic pt

if all the eigenvalues of $DF^n(p)$ are $> 1$ in abs. value

3. $p$ is a *saddle point* otherwise.

*Theorem 6.3* Suppose $F$ has an attracting fixed point at $p$. Then there is an open set $\ni p$ in which all points tend to $p$ under forward iteration.

Rem. If $W^s(p) = \{ x \mid \lim F^n(x) = p \}$ then $W^s(p)$ is called the *basin of attraction* or the *stable set of $p$*.

Thm 6.3. says that $W^s(p)$ contains an open set about $p$.

*Pf.* With no loss of generality assume $p$ to be $\bar{0}$. $DF(\bar{0})$ is lin. conjugate to one of the standard forms

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad 0 \leq |\lambda|, |\mu| < 1$$

$$\begin{pmatrix} \lambda & \varepsilon \\ 0 & \lambda \end{pmatrix} \qquad \varepsilon > 0 \text{ but arbitrarily small}$$

or
$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \quad 0 < \alpha^2 + \beta^2 < 1.$$

If $\varepsilon > 0$ is chosen small enough we have
$$|DF(\bar{0})\bar{v}| < |\bar{v}| \quad \text{for } \bar{v} \neq \bar{0}$$
(see proof of Prop. 2.7)

By continuity of $DF(\bar{x})$ we get
$$|DF(\bar{x})\bar{v}| < |\bar{v}| \quad \text{for } \bar{x} \text{ in some}$$
nbhd of $\bar{0}$, $U$, say.

Now let $\delta > 0$ be such that
$$|\bar{y}| < \delta \Rightarrow \bar{y} \in U.$$

Claim: $|F(\bar{y})| < |\bar{y}|$ (and so $F^n(\bar{y}) \to \bar{0}$)

Let $\gamma(t) = t \cdot \bar{y}$, $0 \le t \le 1$.

Then $\gamma(0) = \bar{0}$, $\gamma(1) = \bar{y}$

$F(\gamma(0)) = \bar{0}$, $F(\gamma(1)) = F(\bar{y})$ and

$\gamma(t) \in U$, $0 \le t \le 1$.

$$|F(\bar{y})| = \left| \int_0^1 F(\gamma(t))' \, dt \right|$$
$$\le \int_0^1 |F(\gamma(t))'| \, dt$$
$$= \int_0^1 |DF(\gamma(t)) \cdot \gamma'(t)| \, dt$$

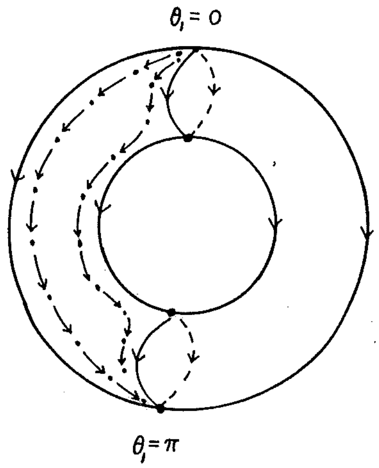



**Fig. 6.3.** The phase portrait of $F$ on the torus.

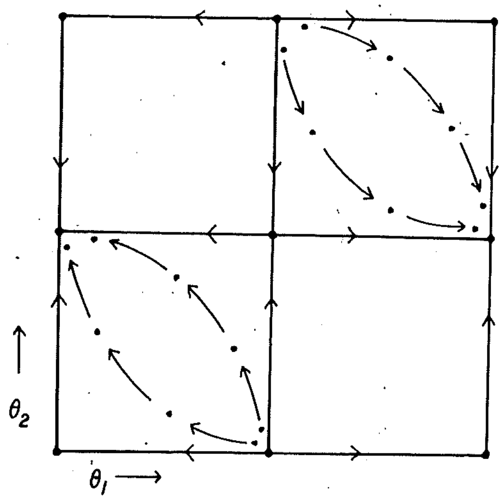



**Fig. 6.4.** The phase portrait of $G\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}\theta_1-\epsilon\sin\theta_1\\ \theta_2+\epsilon\sin\theta_2\end{pmatrix}$.

In this case, the unstable manifolds of both saddles tend to the sink, while their stable manifolds come from the source.

The behavior of the stable and unstable manifolds play an important role in the question of the structural stability of a higher dimensional dynamical

$$< \int_0^1 |\gamma'(t)|\, dt = \int_0^1 |\bar{y}|\, dt = |\bar{y}|.$$

$$\therefore \quad |F(\bar{y})| < |\bar{y}| .$$

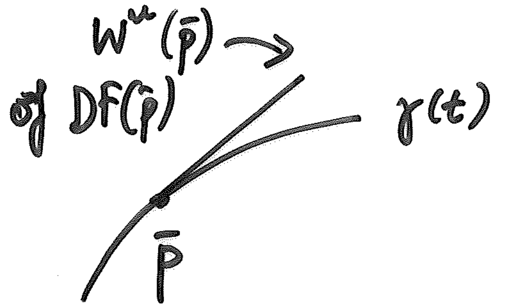
<u>Theorem 6.5.</u> Suppose $F$ has a saddle point at $\bar{p}$. There exists $\varepsilon > 0$ and a smooth curve (a $C^1$ curve)

$$\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$$

such that

1. $\gamma(0) = \bar{p}$
2. $\gamma'(t) \neq 0$
3. $\gamma'(0)$ is an unstable eigenvector for $DF(\bar{p})$ (i.e. eigenvalue is $> 1$ in abs. value)
4. $\gamma$ is $F^{-1}$-invariant
5. $F^{-n}(\gamma(t)) \to \bar{p}$ as $n \to \infty$
6. If $|F^{-n}(\bar{q}) - \bar{p}| < \varepsilon$ for all $n \geq 0$ then $q = \gamma(t)$ for some $t$

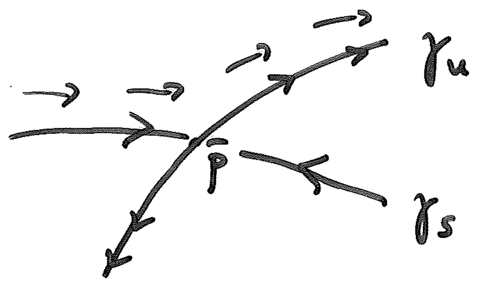
The curve is the <u>local</u> <u>unstable</u> <u>manifold</u> <u>at $\bar{p}$</u>




Remarks.

1. Call the curve $\gamma_u$.

With obvious modifications we can assert the corresponding things about <u>the local stable manifold</u> $\gamma_s$. Here $F^n(\gamma_s(t)) \to \bar{p}$ etc.




2. Higher dimensions: local stable/unstable manifold is a surface (or a higher dim. object) dep. on the no. of eigenvalues of $DF(\bar{p}) > 1$ or $< 1$.

3. $\gamma$ is $C^\infty$.