Cayley - Hamilton Theorem

$$p(A) = 0$$

Pf. Case $n = 2$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad A^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ca + dc & cb + d^2 \end{pmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \det \begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - (a+d)\lambda - bc + ad$$

$$p(A) = \begin{pmatrix} a^2 + bc - (a+d)a - bc + ad & ab + bd - (a+d)b \\ ca + cd - (a+d)c & cb + d^2 - (a+d)d - bc + ad \end{pmatrix}$$

$$\|$$

$$A^2 - (a+d)A + (ad - bc)I$$

$$\|$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Arthur Cayley 1821 - 1895 Cambridge

Sir William Rowan Hamilton 1805 - 1865 Dublin

$\exists\ \bar{v}_1, \bar{v}_2$ lin. indep. so that $G = (\bar{v}_1, \bar{v}_2)$ gives case 3. Also $\lambda \neq \mu$ gives case 3.

<u>Prop 1.12</u> The linear map

$$L(\bar{x}) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \bar{x}$$

is linearly conjugate to

$$L_\epsilon(\bar{x}) = \begin{pmatrix} \lambda & \epsilon & 0 \\ 0 & \lambda & \epsilon \\ 0 & 0 & \lambda \end{pmatrix}$$

for any $\epsilon \neq 0$.

2-dim: $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \sim \begin{pmatrix} \lambda & \epsilon \\ 0 & \lambda \end{pmatrix}$

<u>Pf.</u> Put $S_\epsilon(\bar{x}) = \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Look at $S \circ L \circ S^{-1}$.

Def 1.13
$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & 1 \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}$$
is called a <u>Jordan</u> <u>block</u>.

Def 1.14 A matrix A is in <u>Jordan</u> <u>form</u>

if $$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}$$

where A_i, $1 \le i \le k$, is a Jordan block.

- For any A, $\exists$ <u>complex</u> G so that $G^{-1} A G$ is in Jordan form

- λ may be complex
- Diagonal entries are the eigenvalues
- No. of λ's = algebraic multiplicity of λ

  No of blocks with λ as eigenvalue = geometric multiplicity of λ

Camille Jordan 1838 – 1922

The complex λ's come in pairs $\lambda, \bar{\lambda}$.
They may be regrouped into a
<u>real Jordan form</u> with blocks

$$\begin{pmatrix} A & I & O & \cdots & O \\ O & A & I & \cdots & O \\ O & O & A & I \cdots & O \\ & & & \ddots & \\ \cdot & \cdots & \cdots & & A \end{pmatrix}$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$

(where $\alpha + i\beta = \lambda$). G <u>real</u>.

Functions from $R^2 \to R^2$

$F(x, y) \in R^2$

$x_1 = f_1(x, y)$
$y_1 = f_2(x, y)$

Not.

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} = F\begin{pmatrix} x \\ y \end{pmatrix}$$

Jacobian (matrix) of F

$$DF(\bar{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x}(\bar{x}) & \frac{\partial f_1}{\partial y}(\bar{x}) \\ \frac{\partial f_2}{\partial x}(\bar{x}) & \frac{\partial f_2}{\partial y}(\bar{x}) \end{pmatrix}$$

sv.: totala derivatan, funktionalmatrisen,
Jacobimatrisen

$$F(\bar{x}+\bar{h}) = F(\bar{x}) + DF(\bar{x})\cdot\bar{h} + \mathcal{O}(|\bar{h}|^2)$$

cf. 1-dim.

$$f(x+h) = f(x) + f'(x)\cdot h + \mathcal{O}(h^2)$$

Ex.

$$F(\bar{x}) = \begin{pmatrix} ax - bxy \\ dy + cxy \end{pmatrix} \qquad a>1,\ d<1$$

Lotka-Volterra, Predator-Prey Model

Alfred J. Lotka 1880 - 1949 USA
Vito Volterra 1860 - 1940 Rome

$$DF(\bar{x}) = \begin{pmatrix} a - by & -bx \\ cy & d + cx \end{pmatrix}$$

$$DF(\bar{0}) = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$$

$$DF\begin{pmatrix} \frac{1-d}{c} \\ \frac{a-1}{b} \end{pmatrix} = \begin{pmatrix} 1 & -b \cdot \frac{1-d}{c} \\ c \cdot \frac{a-1}{b} & 1 \end{pmatrix}$$

Ex. Hénon map

$$F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a - by - x^2 \\ x \end{pmatrix}$$

$$DF\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x & -b \\ 1 & 0 \end{pmatrix}$$

Michel Hénon (1931-) French astronomer, Nice

<u>Def.</u> F is C^1 if DF exists and is cont's
F is C^∞ if all k^{th} partial deriv. exist, for all k

<u>Def. 1.16</u> $F: \mathbb{R}^2 \to \mathbb{R}^2$ is a <u>diffeomorphism</u> if F is 1-1, onto, C^∞ and F^{-1} is C^∞.

74 **Def 1.18** If X and Y are arbitrary

$$X \times Y = \{ (x, y) \mid x \in X, \ y \in Y \}$$

Ex. $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$

$\mathbb{R} \times S_1$ cylinder

$S_1 \times S_1$ surface of a torus, "doughnut"

$S_1 \times B^2$ solid torus

where $B^2 = \{ \bar{x} \in \mathbb{R}^2 \mid |\bar{x}| \le 1 \}$

Theorem 1.19 The Implicit Function Theorem

Suppose $F: \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$F\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} f_1(x, y, z) \\ f_2(x, y, z) \end{pmatrix}$$

If $F(\bar{0}) = \bar{0}$ and

$$\begin{pmatrix} \frac{\partial f_1}{\partial x}(\bar{0}) & \frac{\partial f_1}{\partial y}(\bar{0}) \\ \frac{\partial f_2}{\partial x}(\bar{0}) & \frac{\partial f_2}{\partial y}(\bar{0}) \end{pmatrix}$$

is non-singular (i.e., invertible), then there exists $\varepsilon > 0$ and a smooth curve $\mathfrak{z}(z)$ of the form

$$x = \mathfrak{z}_1(z), \quad y = \mathfrak{z}_2(z), \quad -\varepsilon < z < \varepsilon$$

such that $F(\mathfrak{z}_1(z), \mathfrak{z}_2(z), z) = 0.$

<u>Theorem 1.20</u> The Inverse Function Theorem

Let $F: \mathbb{R}^2 \to \mathbb{R}^2$. Suppose $F(\bar{0}) = \bar{0}$ and that $DF(\bar{0})$ is invertible. Then there exists a neighborhood U of $\bar{0}$ and a C^∞ map $G: U \to \mathbb{R}^2$ such that

$$F \circ G(\bar{x}) = \bar{x}$$

for all $\bar{x} \in U$.

<u>Theorem 1.21</u> The Contraction Mapping Thm

Let $F: B^2 \to B^2$ where B^2 is the closed unit disk in $\mathbb{R}^2$. Suppose

$$|F(\bar{x}_1) - F(\bar{x}_2)| < \lambda |\bar{x}_1 - \bar{x}_2)$$

for all $\bar{x}_1, \bar{x}_2 \in B^2$ and a $\lambda < 1$. Then there exists a unique fixed point $\bar{X}_*$ for F. Moreover

$$F^n(\bar{x}) \longrightarrow \bar{X}_* \ , \quad \bar{x} \in B^2.$$

Pf as in the 1-dimensional case:

step 1 Take $\bar{x} \in B^2$, form $\{F^n(\bar{x})\}$

step 2 $\{F^n(\bar{x})\}$ is a Cauchy seq.

step 3 $\{F^n(\bar{x})\}$ is convergent, limit is a fixed point X_*

step 4 Fixed point is unique.