Linear Algebra Review

2.1.

$\bar{x} \in \mathbb{R}^n$ or $\bar{x} \in \mathbb{C}^n$

$$\bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i \in \mathbb{R} \text{ or } \mathbb{C}$$

<u>Def. 1.1</u> $L : \mathbb{R}^n \to \mathbb{R}^n$ is <u>linear</u> if
$L(\alpha \bar{v} + \beta \bar{w}) = \alpha L(\bar{v}) + \beta L(\bar{w})$
all $\alpha, \beta \in \mathbb{R}$ and all $\bar{v}, \bar{w} \in \mathbb{R}^n$.

$L(0) = 0$

Let $\bar{e}_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ i^{th} coord. $= 1$. <u>standard basis vectors</u>

Then $\bar{x} = \sum_{i=1}^{n} x_i \bar{e}_i$.

Call A be a <u>matrix</u>

$$(a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}.$$

of dim. $n \times m$.

$$A\bar{x} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 \dots + a_{mm}x_m \end{pmatrix}$$

If $L(\bar{e}_i) = \bar{v}_i$ and

$$A = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \bar{v}_1 & \bar{v}_2 & \cdots & \bar{v}_m \\ \vdots & \vdots & & \vdots \end{pmatrix}$$

($\bar{v}_i$ column vectors in A)

then

$$L(\bar{x}) = A\bar{x}$$

matrix representation of L (in standard basis)

Pf. $L(\bar{x})$

$$= L\left(\sum_{i=1}^{m} \bar{e}_i x_i\right) \underset{\text{lin.}}{=} \sum_{i=1}^{m} x_i L(\bar{e}_i)$$

$$= \sum_{i=1}^{m} x_i \bar{v}_i$$

$$(L\bar{x})_j = \sum_{i=1}^{m} x_i (\bar{v}_i)_j = x_1(\bar{v}_1)_j + x_2(\bar{v}_2)_j + \dots + x_m(\bar{v}_m)_j$$

$$= a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jm}x_m$$

$$= (A\bar{x})_j$$

<u>Prop 1.4.</u> Let L and P be linear maps with matrix repr. A and B, respectively. Then

$$\underset{\substack{\uparrow \\ \text{function composition}}}{(P \circ L)}(\bar{v}) = \underset{\substack{\uparrow \\ \text{matrix mult.}}}{(BA)}\,\bar{v}, \quad \bar{v} \in \mathbb{R}^n$$

$$I_n \underset{\substack{\uparrow \\ \text{not.}}}{=} \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & & 1 \end{bmatrix} \qquad n \times n \text{ identity matrix}$$

L is <u>invertible</u> iff L is bijective from $\mathbb{R}^n$ to $\mathbb{R}^n$

iff its matrix repr. A has an <u>inverse</u> B (i.e., $BA = AB = I_n$). Not.: A^{-1}.

Note: A is invertible $\Longleftrightarrow$ $\det(A) \neq 0$.

<u>Def. 1.5</u> L_1 and L_2 linear. They are <u>linearly conjugate</u> if there is a linear, invertible P such that

$$P \circ L_1 = L_2 \circ P$$

or

$$L_1 = P^{-1} \circ L_2 \circ P$$

If $L_1(\bar{x}) = A_1 \bar{x}$, $L_2(\bar{x}) = A_2 \bar{x}$ and $P(\bar{x}) = G\bar{x}$, where A_1, A_2, G are $n \times n$ matrices, then

$$G A_1 = A_2 G \quad \text{or} \quad A_1 = G^{-1} A_2 G.$$

We say that the matrices A_1 and A_2 are <u>similar</u>.

<u>Def. 1.6</u> A $n \times n$ matrix. $\lambda \in \mathbb{C}$ is an <u>eigenvalue</u> of A if it is a root zero of the <u>characteristic</u> <u>polynomial</u> $p(\lambda)$ of A.

$$p(\lambda) = \det(A - \lambda I)$$

An eigenvector $\bar{v}$ associated with λ is a vector $\neq \bar{0}$ such that

$$A\bar{v} = \lambda \bar{v}$$

The algebraic multiplicity of λ is its multiplicity as a root or zero of $p(\lambda)$.

<u>Note</u> If $A : \mathbb{R}^n \to \mathbb{R}^n$ then it may be extended to $A : \mathbb{C}^n \to \mathbb{C}^n$ by defining

$A(i\bar{y}) = i A\bar{y}$.

The eigenvector & eigenvalue may be <u>complex</u>!

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For A real, $p(\lambda)$ is a real nth degree polynomial. So if λ is a zero then $\bar{\lambda}$ is one, too.

Ex. $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ has eigenvalues $\pm i$

The eigenvector corresp ~~to~~ i is any multiple $\neq 0$ of $\begin{pmatrix} 1 \\ i \end{pmatrix}$.

<u>Prop 1.9</u> If $L_1(\bar{x}) = A_1\bar{x}$ and $L_2(\bar{x}) = A_2\bar{x}$ are linearly conjugate, then A_1 and A_2 have the same eigenvalues.

Pf. $\det(A_1 - \lambda I) = \det(G^{-1}A_2 G - \lambda I)$

$= \det(G^{-1}A_2 G - G^{-1}\lambda G)$

$= \det(G^{-1}(A_2 - \lambda I)G)$

$= \underbrace{\det G^{-1}}_{(\det G)^{-1}} \cdot \det(A_2 - \lambda I) \cdot \det G$

$= \det(A_2 - \lambda I)$

<u>Theorem 1.10</u> Let $L: \mathbb{R}^3 \to \mathbb{R}^3$ be linear with $L(\bar{x}) = A\bar{x}$. There exists a real, invertible 3×3 matrix G such that $G^{-1}AG$ takes one of the four forms

1. $\begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \lambda \end{pmatrix}$

2. $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \eta \end{pmatrix}$

3. $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}$

4. $\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$

where all entries are real, and $\beta \neq 0$.

Called: <u>standard forms</u>.

<u>Cor 1.11</u> Let A be 2×2 real. Then there exists a real invertible 2×2 matrix G such that $G^{-1}AG$ is of one of the forms

1. $\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$

2. $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

3. $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$

<u>Proof</u> of the 2×2 case

First, let A have complex eigenvalues

$\lambda_+ = \alpha + i\beta$, $\lambda_- = \alpha - i\beta$.

Let $\bar{w}$ be the eigenvector associated with $\alpha + i\beta$.

Write $\bar{w} = \bar{v}_1 + i\bar{v}_2$, $\bar{v}_i$ real 2-dim. vectors.

Since $A\bar{w} = (\alpha + i\beta)\bar{w} = (\alpha + i\beta)(\bar{v}_1 + i v_2)$

$= \alpha\bar{v}_1 - \beta\bar{v}_2 + i(\beta\bar{v}_1 + \alpha\bar{v}_2)$

$\bar{v}_1, \bar{v}_2$ lin. indep.

$A\bar{v}_1 = \alpha\bar{v}_1 - \beta\bar{v}_2$

$A\bar{v}_2 = \beta\bar{v}_1 + \alpha\bar{v}_2$

$$G = \begin{pmatrix} \bar v_1 & \bar v_2 \end{pmatrix}$$ is invertible since $\bar v_1, \bar v_2$ lin. indep.

Check

$$G \begin{pmatrix} \alpha & +\beta \\ -\beta & \alpha \end{pmatrix} = AG.$$

$$\begin{pmatrix} (\bar v_1)_1 & (\bar v_2)_1 \\ (\bar v_1)_2 & (\bar v_2)_2 \end{pmatrix} \begin{pmatrix} \alpha & +\beta \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} (\bar v_1)_1\alpha - (\bar v_2)_1\beta & +(\bar v_1)_1\beta + (\bar v_2)_1\alpha \\ (\bar v_1)_2\alpha - (\bar v_2)_2\beta & +(\bar v_1)_2\beta + (\bar v_2)_2\alpha \end{pmatrix}$$

$$AG = \begin{pmatrix} A\bar v_1 & A\bar v_2 \end{pmatrix} = \begin{pmatrix} (\bar v_1)_1\alpha - (\bar v_2)_1\beta & (\bar v_1)_1\beta + (\bar v_2)_1\alpha \\ (\bar v_1)_2\alpha - (\bar v_2)_2\beta & (\bar v_1)_2\beta + (\bar v_2)_2\alpha \end{pmatrix}$$

Secondly, if one eigenvalue λ is real then so is the other μ.

CASE 2.

If $\lambda = \mu$ we can have

$(A - \lambda I)^2 \bar v = 0$ for a $\bar v \neq 0$ but cf. 68a

(*) $(A - \lambda I)\bar v \neq 0$

Take $\bar w_1 = \bar v$ and $(A - \lambda I)\bar w_1 = \bar w_2$. Then

$A\bar w_2 = \lambda \bar w_2$ (because $(A-\lambda I)^2 \bar w_1 = 0$)

$A\bar w_1 = \lambda \bar w_1 + \bar w_2$ (def. of $\bar w_2$)

$\bar w_1, \bar w_2$ lin. indep.

Check again that $G = \begin{pmatrix} \bar w_2 & \bar w_1 \end{pmatrix}$ works.

If, in the preceding (*) $(A - \lambda I)\bar v = 0$ for all $\bar v$, then the nullspace of $A - \lambda I$ is two-dim.