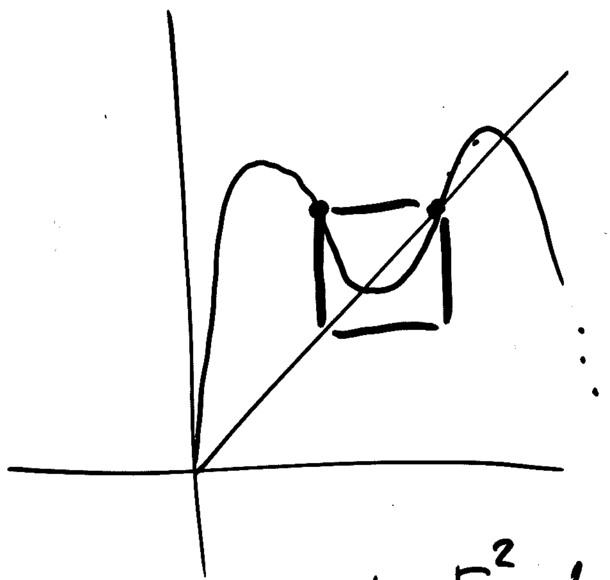
 Sections 1.13 – 1.16 skipped.

1.17 The period-doubling route to chaos

Consider $F_\mu(x) = \mu x(1-x)$ for $\mu > 1$. It has fixed point $p_\mu = \frac{\mu - 1}{\mu}$ and (in the case of $\mu > 2$) it has a partner $\hat{p}_\mu = \frac{1}{\mu}$ with the property of $F_\mu(\hat{p}_\mu) = p_\mu$.
If we now draw F_μ^2 and consider the quadratic "window" defined by $\hat{p}_\mu$ and p_μ we observe that the graph looks like a logistic map in a different scale and upside down.



Clearly $F_\mu^2(\hat{p}_\mu) = p_\mu$

Furthermore if F_μ^2 has a fixed point $\neq p_\mu$ in this window, then that is a 2-period point for the original system.

56 The window enlargement is done by the following linear transformation

$$L_\mu(x) = \frac{x - p_\mu}{\hat{p}_\mu - p_\mu}$$

with inverse

$$L_\mu^{-1}(x) = (\hat{p}_\mu - p_\mu)\,x + p_\mu .$$

(Note that the denominator in L_μ is negative.)

The <u>renormalization</u> <u>operator</u> is defined as

$$(R F_\mu)(x) = L_\mu \circ F_\mu^2 \circ L_\mu^{-1}(x)$$

We have

- $RF_\mu(0) = 0$, $\quad RF_\mu(1) = 0$
- $(RF_\mu)'(\tfrac{1}{2}) = 0$ and $\tfrac{1}{2}$ is the only critical point of RF_μ
- If $RF_\mu(q) = q$ then F_μ has a point of period 2.
- RF_μ is not defined unless $F_\mu'(p_\mu)$ is negative, because then we don't get a corresponding point $\hat{p}_\mu$ with $F_\mu(\hat{p}_\mu) = p_\mu$, $\hat{p}_\mu < p_\mu$.

57 For certain μ the R-operator can be applied <u>many</u> <u>times</u>. Such F_μ have periods 2, 4, or higher.

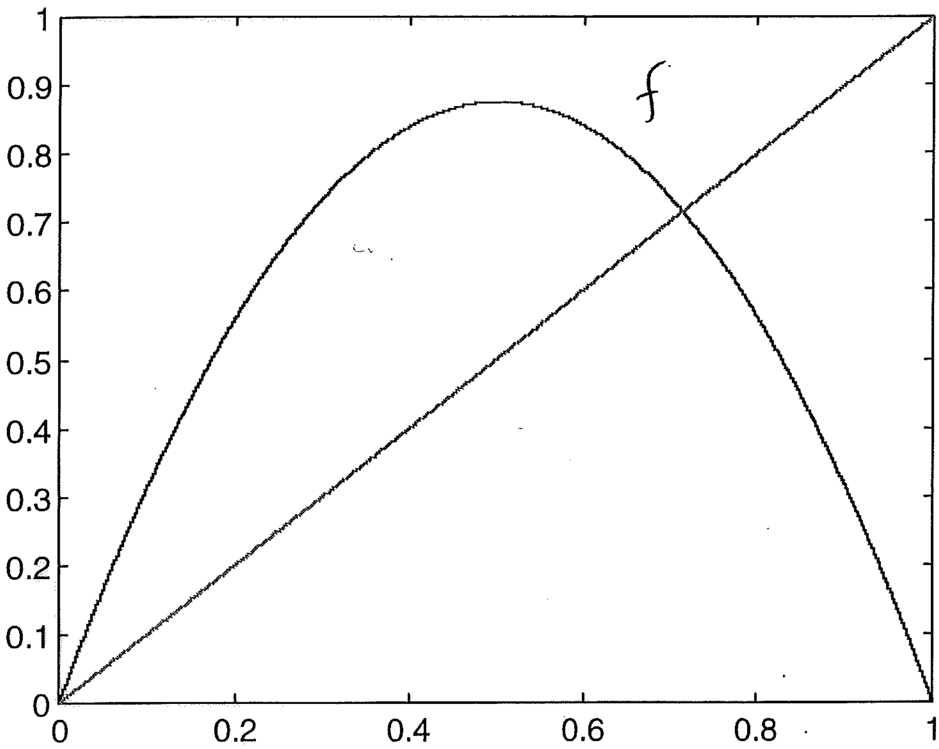
One can show that for $\mu^* = 3,569945672\ldots$ we can renormalize F_μ infinitely many times, i.e. F_μ has periodic points of prime periods 2^n for all n.

For $Q_c = x^2 + c$ the critical value of c is $-1,401155189\ldots$, which is called the Myrberg point (P. J. Myrberg, Finnish mathematician, 1892 - 1976).
(Recall that $F_\mu \sim Q_c$.)

Hence when μ increases from 3 to μ^* F_μ undergoes period-doublings, from period 2 to 4 to 8 to 16 ...

```
t=0:.0001:1;
m=3.5;
f=m*t.*(1-t);
plot(t,f,t,t)
```

$F_{3.5}$



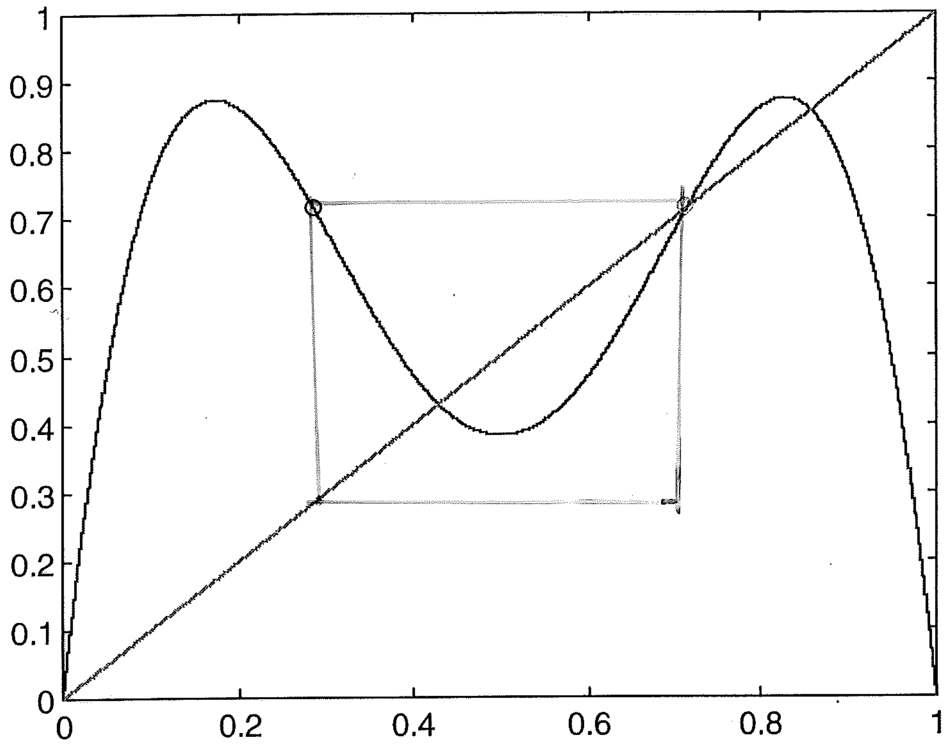

```
p1=1/m,p2=1-p1

p1 =
     0.2857
p2 =
     0.7143

f2=m*f.*(1-f);

plot(t,f2,t,t)
```

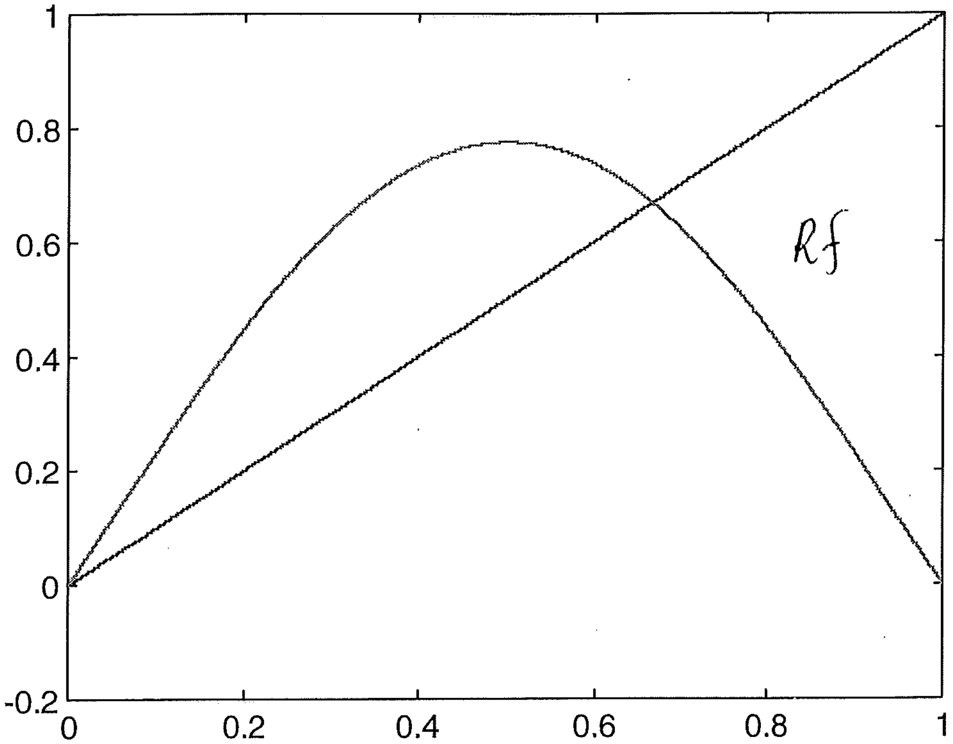
$F_{\mu}^{2} \quad \mu = 3{,}5$



```
for i = 1:10001, Rf(i)=f2(fix(10000*((p1-p2)*.0001*i + p2)))./(p1-
p2)+p2/(p2-p1);end;

plot(t,t,t,Rf)
```

59 a

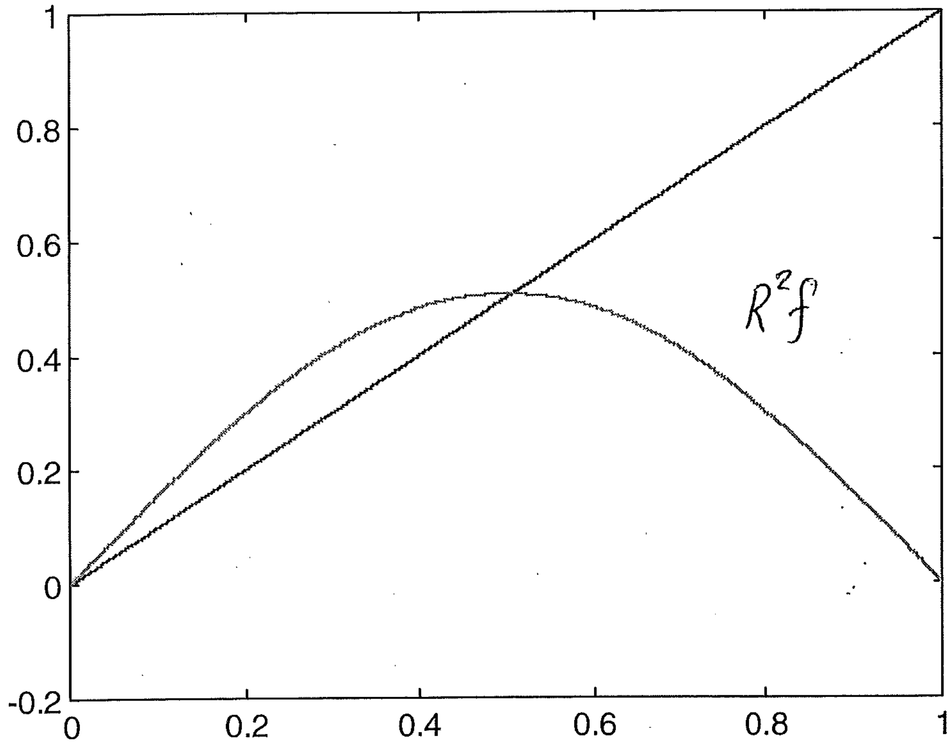



```
for i=1:10001, R2(i)=Rf(fix(10000*abs(Rf(i)))+1);end; plot(t,t,t,R2)
```

$f = F_{3.5}$

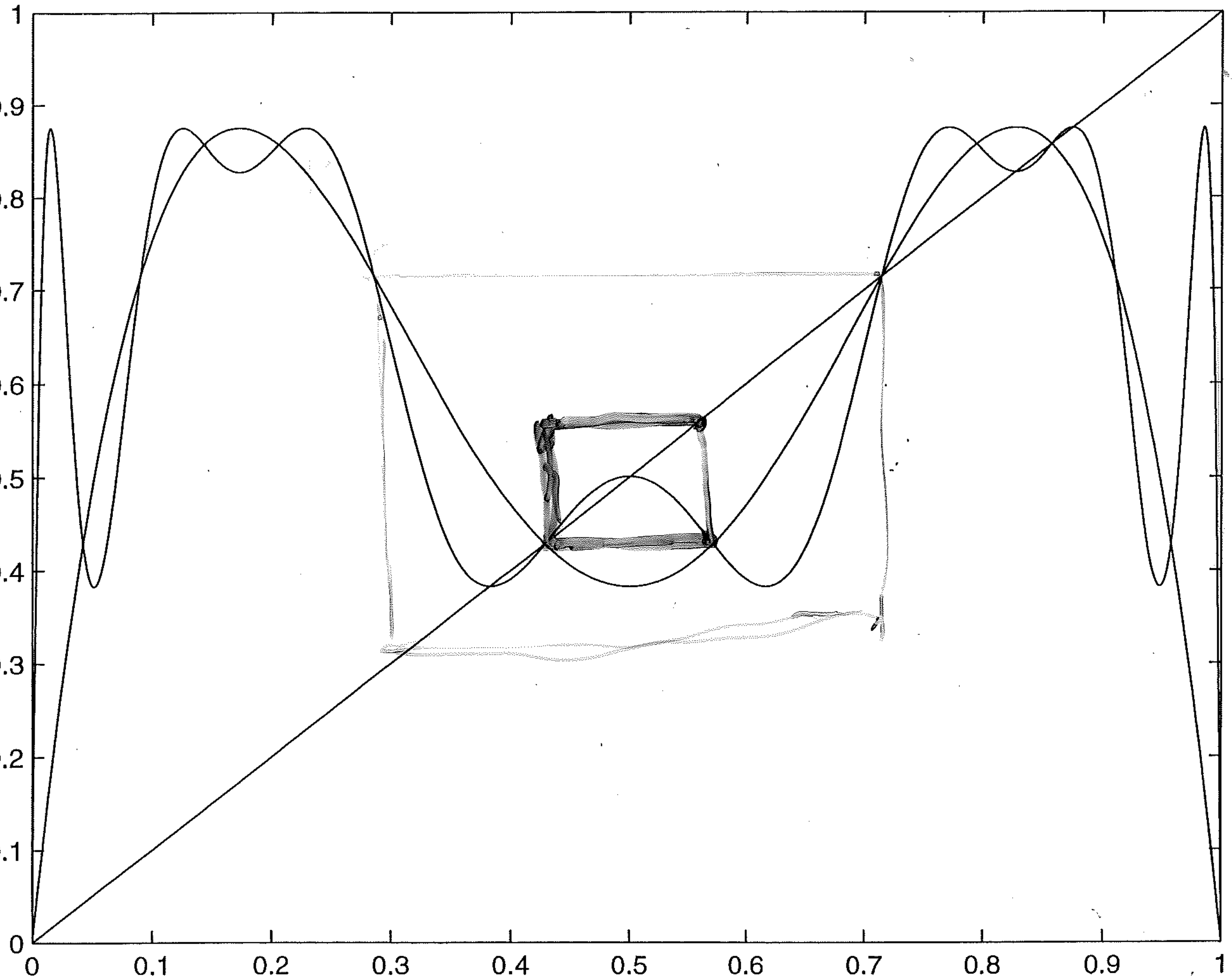
59 b

Second renormalization R2f using the same principles as for Rf:



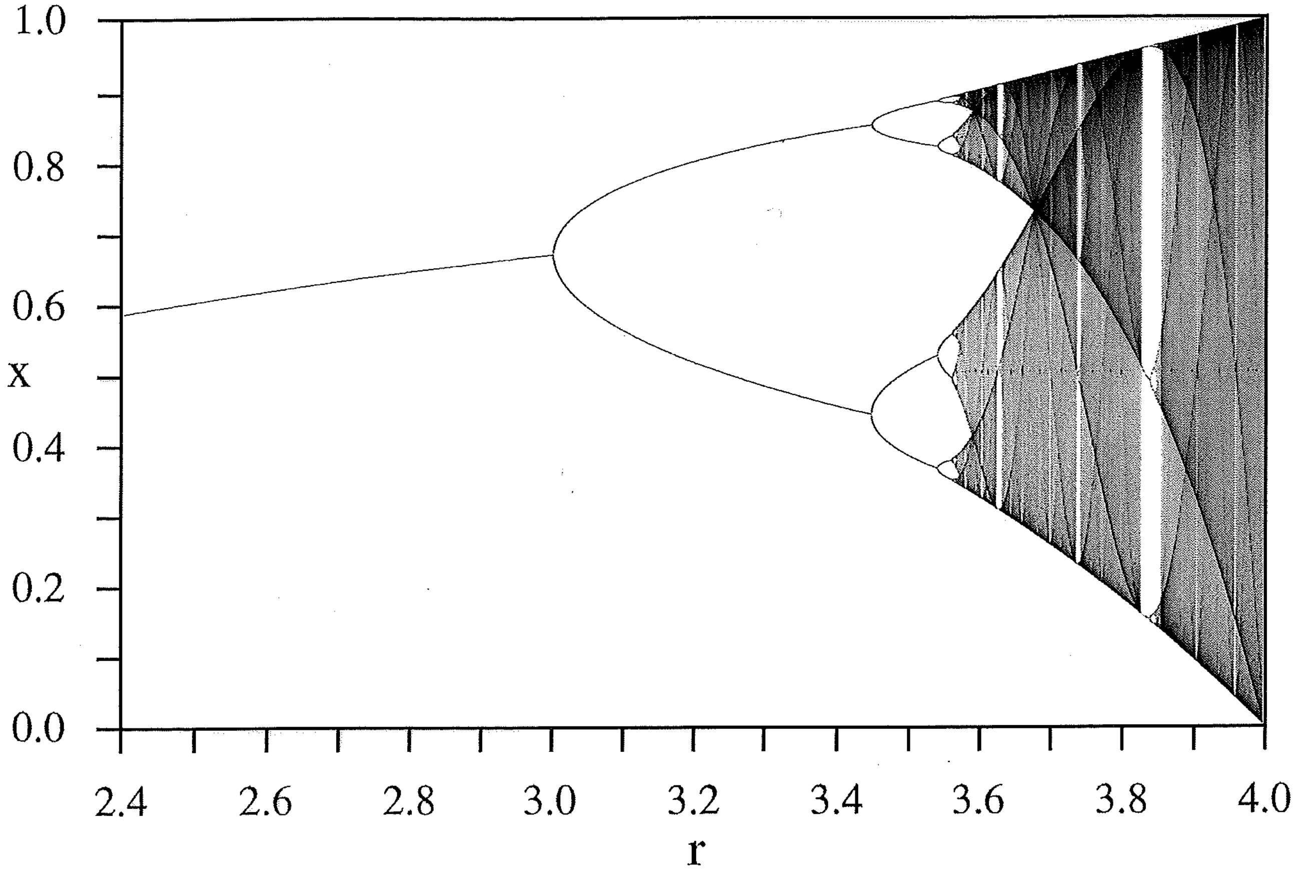
$f = F_{3.5}$

59c



$F_{3.5}(x) = 3.5 \cdot x \cdot (1-x)$

$F_{3.5}^2$ and $F_{3.5}^4$


1.0
0.8
0.6
x
0.4
0.2
0.0
2.4
2.6
2.8
3.0
3.2
3.4
3.6
3.8
4.0
r

