1.12. Bifurcation Theory

1.11. Not covered

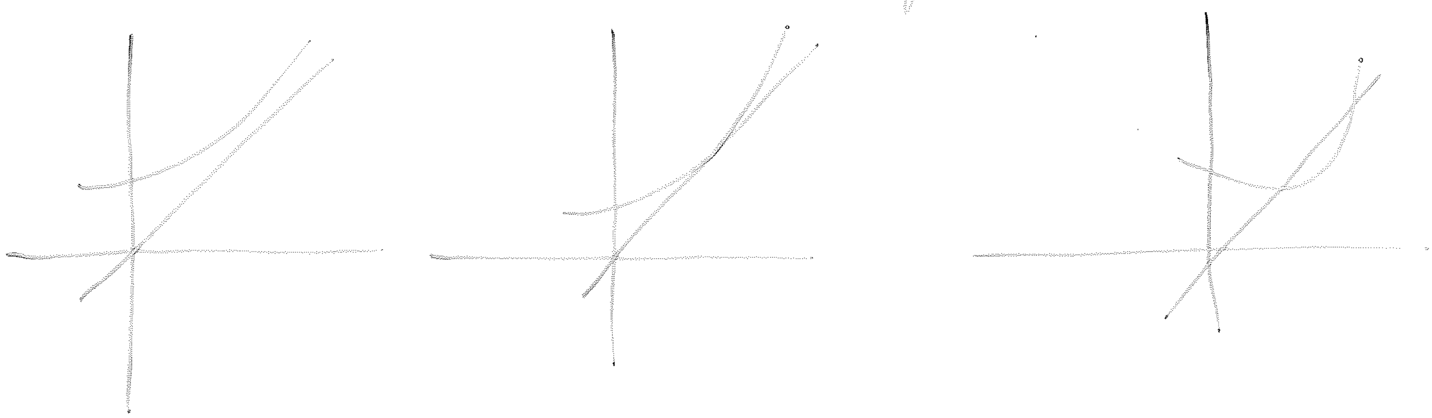
$$G(x,\lambda) = f_\lambda(x) \quad \text{as } \lambda \to \lambda_0$$

Ex.

$$F_\mu(x) = \mu x(1-x) \quad \text{as } u \to 3$$

Ex. 12.1.

The Saddle-Node or Tangent Bifurcation



Ex. $Q(x) = x^2 + c$ when $c > \frac{1}{4}$, $c = \frac{1}{4}$, $c < \frac{1}{4}$

$E_\lambda(x) = \lambda e^x$ $\quad \lambda > 1/e$, $\lambda = 1/e$, $0 < \lambda < 1/e$

$\lambda e^x = x$ has no solution when $\lambda > \frac{1}{e}$

one $\quad \lambda = \frac{1}{e}$

two $\quad 0 < \lambda < 1/e$

Analysis: $\lambda e^x - x$ has its minimum at the point x_0 where $\lambda e^x - 1 = 0$, $x_0 = \log \frac{1}{\lambda}$

Value is $\lambda \cdot \frac{1}{\lambda} - \log\frac{1}{\lambda} = 1 + \log \lambda < 0$ iff $\lambda < \frac{1}{e}$.

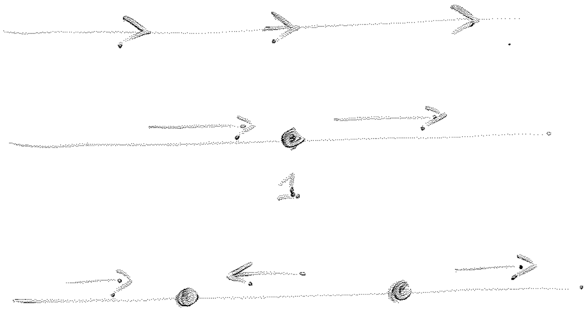
Phase portrait of E_λ

$\lambda > \frac{1}{e}$ — no fixed point

$\lambda = \frac{1}{e}$ — one —"—

1

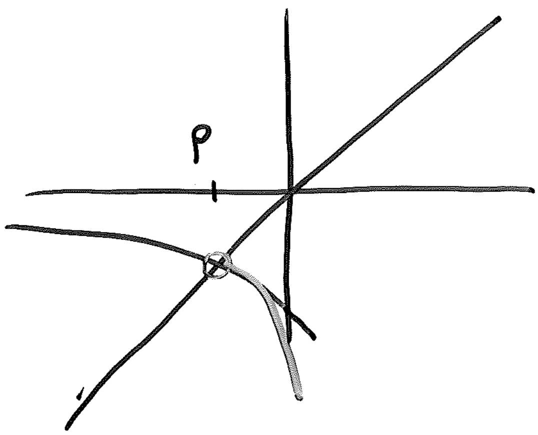
$\lambda < \frac{1}{e}$ — two —"—s



Period-Doubling Bifurcation

F_μ at $\mu = 3$

$E_\lambda(x)$ $\lambda < 0$



p

$\underline{0 > \lambda > -e}$

$E_\lambda(x) - x = \lambda e^x - x = 0$

has one root at p, say.

Value of multiplier is λe^p, i.e.

$= p$.

Since $E_\lambda(-1) = \frac{\lambda}{e}$ is > -1, $p \in (-1, 0)$

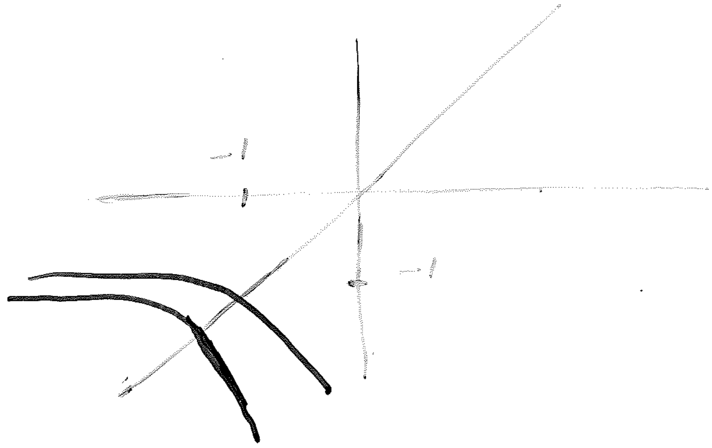
Thus $-e < \lambda < 0 \Rightarrow$ attractive fixed point in $(-1, 0)$

$\underline{\lambda = -e}$

Non-hyperbolic fixed point at $p = -1$, mult. $= -1$.

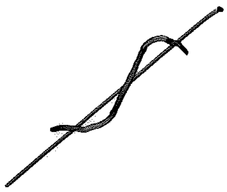
$\underline{\lambda < -e}$

Hyperbolic fixed point at $p < -1$, repelling



$E_\lambda'(p) = -1$ for $\lambda = -e$; $(E_\lambda^2)'(p) = 1$

< -1 $\quad \lambda < -e$; $(E_\lambda^2)'(p) > 1$



$E_\lambda^2 \quad \lambda < -e$

The period-doubling bifurcation

1. the attracting fixed point becomes repelling
2. a new attracting 2-period is born

Ex. 12.3.

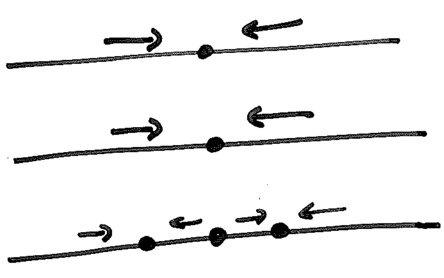
$S(x) = \lambda \sin x$ $\qquad S'(0) = \lambda$

Fixed point at 0

attr. $\quad 0 < \lambda < 1$

non-hyp. $\quad \lambda = 1$

repell. $\quad 1+\varepsilon > \lambda > 1$



0

$f_\lambda \in C^1$

<u>Theorem 12.5</u> $\{f_\lambda\}$ one-parameter family of functions. Assume

$$f_{\lambda_0}(x_0) = x_0$$
$$f'_{\lambda_0}(x_0) \neq 1$$

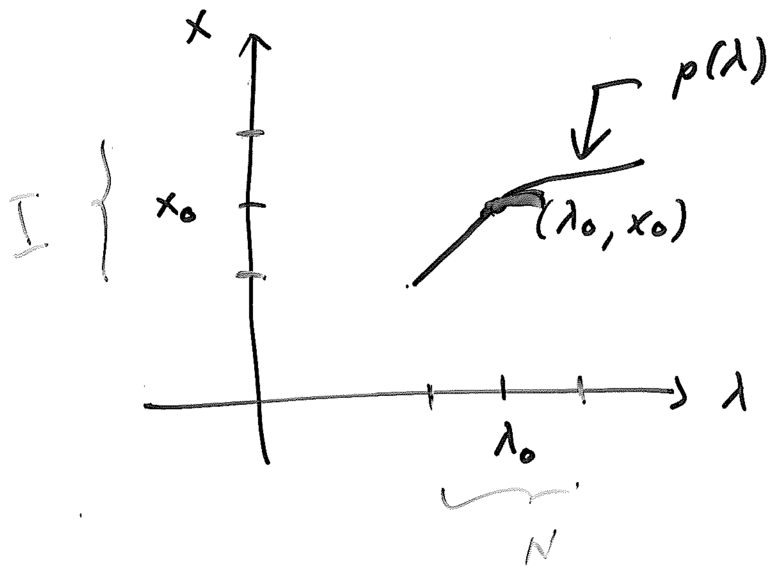
Then there are intervals I about x_0 and N about λ_0 and a smooth function $p : N \to I$ such that

$$p(\lambda_0) = x_0$$

and

$$f_\lambda(p(\lambda)) = p(\lambda)$$

Moreover, f_λ has no other fixed point in I.



<u>Pf.</u> $G(x,\lambda) = f_\lambda(x) - x$. We wish to use the Implicit Function Theorem to write

$$\{x \mid G(x,\lambda) = 0\}$$

as a function (suitably restr. in nbhd of (x_0, λ_0).)

$$G(x_0, \lambda_0) = 0 \quad \text{and}$$

$$\frac{\partial G}{\partial x}(x_0, \lambda_0) = f'_{\lambda_0}(x_0) - 1 \neq 0.$$

Then the Implicit Function Theorems tells us that there is a $I \times N \ni (x_0, \lambda_0)$ where $G(x, \lambda) = 0$ defines x as a fct of λ.
More precisely, $\exists$ fct $p(\lambda)$, $\lambda \in N$, $p(\lambda) \in I$ such that

$$G(p(\lambda), \lambda) = 0.$$

$p(\lambda_0) = x_0$ and p differentiable at λ_0.

<u>Note</u> We can use p to "move the fixed points to 0":

Consider $\quad g_\lambda(z) = f_\lambda(z + p(\lambda)) - p(\lambda)$
where z is in a neighborhood of 0, and $z + p(\lambda)$ is in a nbhd of $p(\lambda_0)$.
Then $g_\lambda(0) = f_\lambda(p(\lambda)) - p(\lambda) = 0$,
all $\lambda \in N$.

Note that $g_\lambda \sim f_\lambda$ since

$$g_\lambda(z) + p(\lambda) = f_\lambda(z + p(\lambda))$$

$$h_\lambda \circ g_\lambda = f_\lambda \circ h_\lambda$$

where $h_\lambda(x) = x + p(\lambda)$.

Theorem 12.6. (The saddle-node bifurcation)

Suppose that

1. $f_{\lambda_0}(0) = 0$

2. $f'_{\lambda_0}(0) = 1$

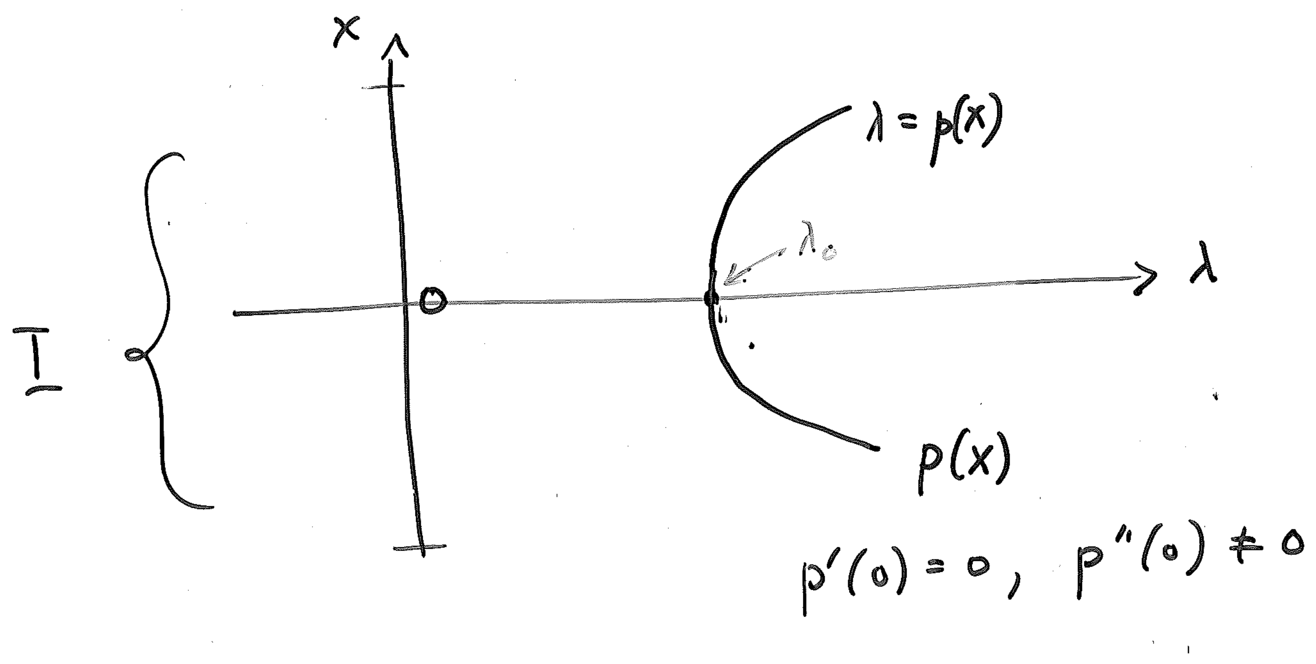
3. $f''_{\lambda_0}(0) \neq 0$

4. $\left.\dfrac{\partial f_\lambda}{\partial \lambda}\right|_{\lambda=\lambda_0} (0) \neq 0$

Then $\exists$ interval $I \ni 0$ and a smooth function $p : I \to \mathbb{R}$ with $p(0) = \lambda_0$ and

$$f_{p(x)}(x) = x.$$

Moreover, $p'(0) = 0$, $p''(0) \neq 0$.



Pf: Let $G(x,\lambda) = f_\lambda(x) - x$.

Then $G(0,\lambda_0) = 0$.

In general, if $G(x,\lambda)=0$ then f_λ has a fixed point at x.

$$\frac{\partial G}{\partial \lambda}(0,\lambda_0) = \left.\frac{\partial f_\lambda}{\partial \lambda}\right|_{\lambda=\lambda_0}(0) \underset{4.}{\underset{\uparrow}{\neq}} 0$$

Again, exists a smooth $p(x)$ with $p(0)=\lambda_0$, and $G(x, p(x)) = 0$. Implicit differentiation:

$$\frac{\partial G}{\partial x} + \frac{\partial G}{\partial \lambda}\cdot p'(x) = 0$$

$$p'(x) = \frac{-\frac{\partial G}{\partial x}(x,p(x))}{\frac{\partial G}{\partial \lambda}(x,p(x))} \quad \leftarrow f_\lambda'(x)-1$$

and

$$p''(x) = \frac{\left(-\frac{\partial^2 G}{\partial x^2}(x,p(x))\cdot 1 - \frac{\partial^2 G}{\partial x\,\partial\lambda}(x,p(x))\cdot p'(x)\right)\cdot\frac{\partial G}{\partial \lambda}(x,p(x)) - \frac{\partial G}{\partial x}(x,p(x))\left(\frac{\partial^2 G}{\partial \lambda\,\partial x}(x,p(x)) + \frac{\partial^2 G}{\partial \lambda^2}(x,p(x))\cdot p'(x)\right)}{\left(\frac{\partial G}{\partial \lambda}\right)^2}$$

$x=0$ gives $p(0)=\lambda_0$ and

$$p''(0) = \frac{\left(-f''_{\lambda_0}(0)\cdot 1 - 0\right)\frac{\partial f}{\partial \lambda}\Big|_{\lambda=\lambda_0}(0) - \left(\underbrace{f'_{\lambda_0}(0) - 1}_{=0}\right)\cdot\left(\qquad\right)}{\left(\left(\frac{\partial f}{\partial \lambda}\right)\Big|_{\lambda=\lambda_0}(0)\right)^2}$$

$$= \frac{-f''_{\lambda_0}(0)}{\frac{\partial f}{\partial \lambda}\Big|_{\lambda=\lambda_0}(0)} \neq 0$$

Theorem 12.7. (Period-doubling bifurcation)

Suppose

1. $f_\lambda(0) = 0$ for all $\lambda \in N$, $\lambda_0 \in N$
2. $f'_{\lambda_0}(0) = -1$
3. $\frac{\partial (f_\lambda^2)'}{\partial \lambda}\Big|_{\lambda=\lambda_0}(0) \neq 0$

Then $\exists\, I \ni 0$ and $p : I \to \mathbb{R}$ smooth

$$f_{p(x)}(x) \neq x, \quad x \neq 0.$$

but $f^2_{p(x)}(x) = x$.