1.10. Sharkovsky's Theorem

A.N. Sharkovsky Ukr. Math. J. 1964
T-Y. Li and J.A. Yorke Amer. Math. Monthly 1975

Theorem 10.1 Let $f: R \to R$ be continuous. Suppose f has a periodic point of prime period 3. Then f has periodic points of all other periods.

Sharkovsky ordering of $\mathbb{Z}_+$, denoted by $\rhd$,

$$3 \rhd 5 \rhd 7 \rhd 9 \rhd 11 \rhd \dots \rhd 2\cdot 3 \rhd 2\cdot 5$$
$$\rhd 2\cdot 7 \rhd 2\cdot 9 \rhd 2\cdot 11 \rhd \dots \rhd 2^2\cdot 3 \rhd 2^2\cdot 5$$
$$\rhd 2^2\cdot 7 \rhd \dots \; 2^3\cdot 3 \rhd 2^3\cdot 5 \rhd \dots\dots \; 2^4 \rhd 2^3$$
$$\rhd 2^2 \rhd 2 \rhd 1.$$

This is a linear (or total) ordering of the positive integers! For every $m, n \in \mathbb{Z}_+$ either $m \rhd n$ or $n \rhd m$.

Theorem 10.2 Suppose f has a periodic point with prime period k. If $k \rhd \ell$, in the S. order, then f also has a periodic point with prime period ℓ.

<u>Lemma</u> I, J closed intervals, $I \subset J$, $f(I) \supset J$.

Then $\exists p \in I$: $f(p) = p$.

<u>Pf.</u> $\max_{x \in I} f(x) \geq \max(J) \geq \max(I)$

$\therefore \quad \max_{x \in I} (f(x) - x) \geq 0$.

Similarly, $\min_{x \in I} (f(x) - x) \leq 0$.

$\therefore \quad \exists p \in I$: $f(p) - p = 0$.

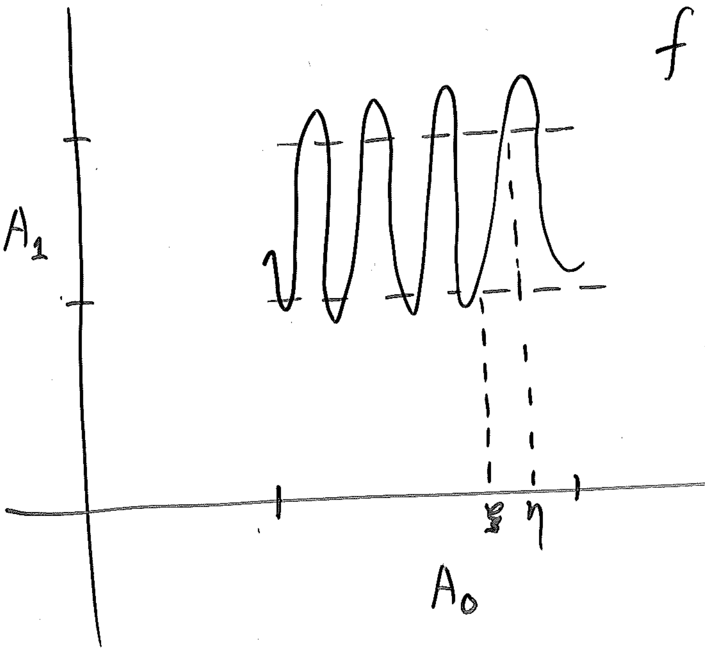
<u>Lemma</u> $f(A_0) \supset A_1$ (A_0, A_1 closed intervals).

Then there is an <u>interval</u> J (closed) such that

$f(J) = A_1$ and $J \subset A_0$.

<u>Pf.</u> Take $\xi \in A_0$ with $f(\xi) =$ (ξ is max. of all poss.) $\min(A_1)$ (exists because $f(A_0) \supset A_1$). Define $\eta = \inf\{x > \xi \mid f(x) = \max A_1\}$

[if $\{\,\} = \phi$ then $\{x < \xi \mid f(x) = \max A_1\}$ is non-empty, take its sup to be η]



Then $[\xi, \eta]$ is mapped onto A_1.

(In the other case, it's $[\eta, \xi]$).

↑ some other ξ

38 **LEMMA:** Suppose $A_0, A_1, \dots, A_n$ are closed intervals such that $f(A_i)$ <u>covers</u> A_{i+1}, meaning

$$f(A_i) \supset A_{i+1}, \quad i = 0, 1, 2, \dots, n-1.$$

Then [Exercise 1, p. 68]

$$\exists x \in A_0 : f(x) \in A_1, f^2(x) \in A_2, \dots, f^{n-1}(x) \in A_{n-1}, f^n(x) \in A_n.$$

<u>Proof of Theorem</u>

Let a, b, c be the 3-cycle, $a < b < c$.

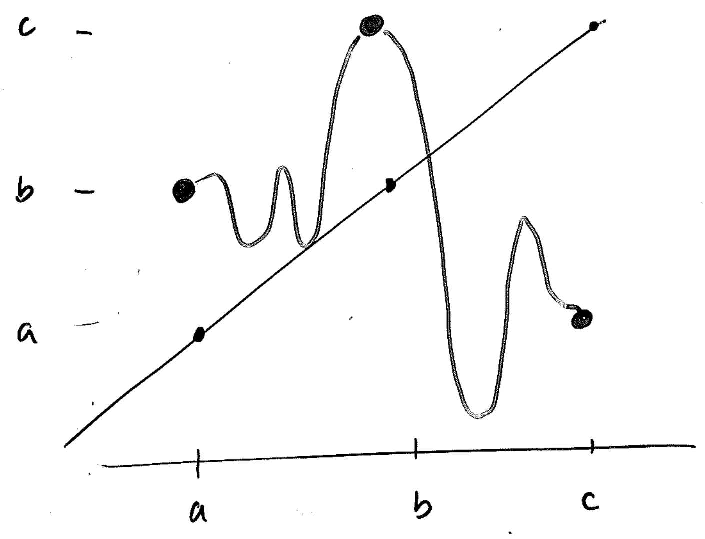
We have either

$$f(a) = b, \ f(b) = c, \ f(c) = a$$

or

$$f(a) = c, \ f(b) = a, \ f(c) = b$$

Look at first case. The second is similar [cf. below]

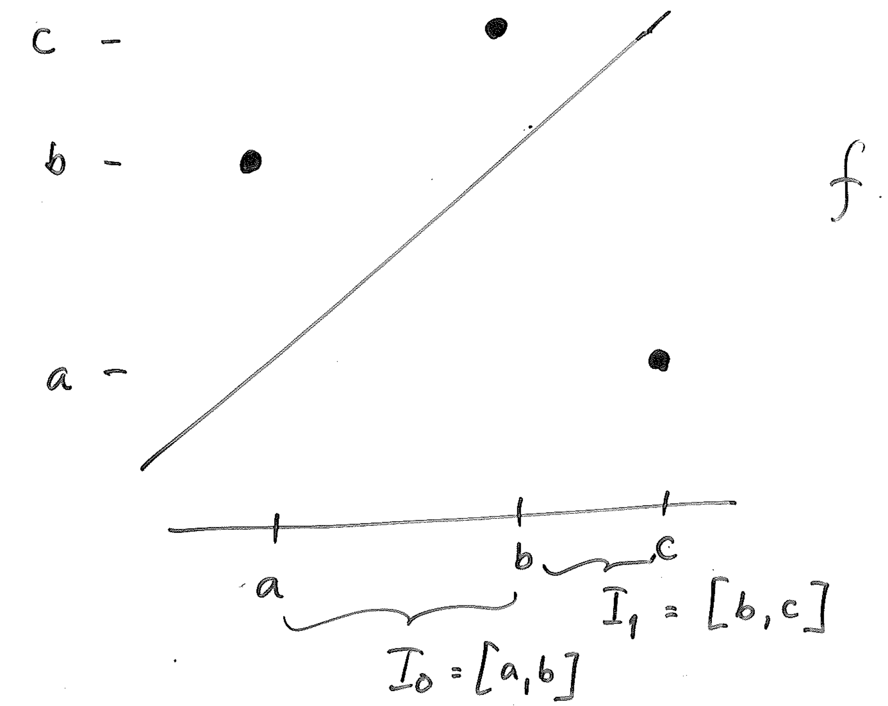


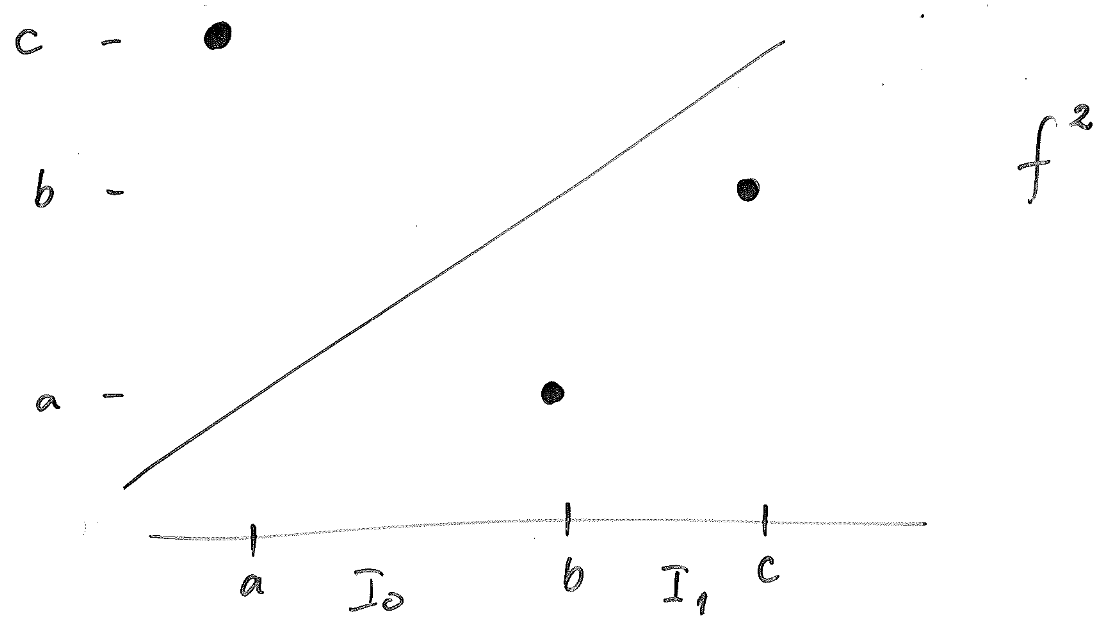
possible f

Remark: In this case $f^2(a) = f(b) = c$

$$f^2(b) = f(c) = a$$

$$f^2(c) = f(a) = b$$





We see that

$$f(I_0) \supset I_1 \quad \text{and} \quad f(I_1) \supset I_0 \cup I_1$$

$\therefore \; \exists\, p \in I_1 : f(p) = p$

Also $\exists\, q \in I_0 : \; f^2(q) = q, \; f(q) \neq q$. Why?

$$f(I_0) \supset I_1, \quad f(I_1) \supset I_0 \cup I_1$$

Can have $q \in$ closed interval J_0 such that

$$f(J_0) = I_1, \quad J_0 \subset I_0, \quad f^2(J_0) \supset I_0 \cup I_1.$$

Then $q \in J_0$, $f(q) \in I_1$, $f^2(q) = q$. [$q \neq b$, of course]

 Now we proceed to construct a point with prime period $n > 3$. Take n arbitrary but > 3.

Define inductively a sequence of nested intervals $\subset I_1$ as follows

$$A_0 = I_1$$

We know that $f(I_1) \supset I_1$. Hence there is a closed interval $A_1 \subset A_0$ with $f(A_1) = I_1$.

Next we find an interval A_2 with $f^2(A_2) = I_1$:

$$f(I_1) \supset I_1 \Rightarrow f^2(A_1) \supset f(I_1) \supset I_1.$$

[Just take A_2 such that $f(A_2) = A_1$].

If $A_3 \subset A_2$ is a subinterval with $f(A_3) = A_2$, then

$$f^3(A_3) = f^2(A_2) = f(A_1) = A_0 = I_1.$$

Continue in this fashion to construct closed intervals

$$A_0 \supset A_1 \supset A_2 \supset \cdots \supset A_{n-2}$$

with $f^i(A_i) = I_1, \ i = 1, 2, \ldots n-2.$

A_{n-1} is taken to satisfy $A_{n-1} \subset A_{n-2}$

$$f^{n-1}(A_{n-1}) = I_0$$

[This works fine since $f^{n-1}(A_{n-2}) = f(I_1) \supset I_0 \cup I_1$]

Also, $f^n(A_{n-1}) = f(I_0) \supset I_1 \supset A_{n-1}$

so

$$f^n(A_{n-1}) \supset A_{n-1}$$

41 $\exists p \in A_{n-1}$ with $f^n(p) = p$.

Claim: p has prime period n.

Proof of claim: By construction

$$p, f(p), \dots, f^{n-2}(p) \in I_1$$
$$f^{n-1}(p) \in I_0$$

If $f^{n-1}(p)$ is an interior point of I_0 then the orbit of p has length $\geq n$, i.e. $= n$.

If $f^{n-1}(p)$ is a border point (a or b) then its orbit has length 3. Then n was 2 or 3. $\lightning$

$\square$

Why: If $f^{n-1}(p) = b$ then

$p = c$ and $f(p) = a \notin I_1$

so $n - 1 = 1$ or $n = 2$

If $f^{n-1}(p) = a$ then

$p = b$, $f(p) = c$, $f^2(p) = a \notin I_1$ $\therefore$ $n - 1 = 2$.

 Theorem 11.2. is not proved here.

Remarks: 1. If there is a period $\neq 2^n$, then there are infinitely many periodic points

2. Result is <u>one-dimensional</u>. Not true even for S^1. (Ex.: Rotation 120°)

3. Converse is also true: $\exists f$ such that f has prime period 5 but <u>not</u> prime period 3

Ex. $f(1) = 3$

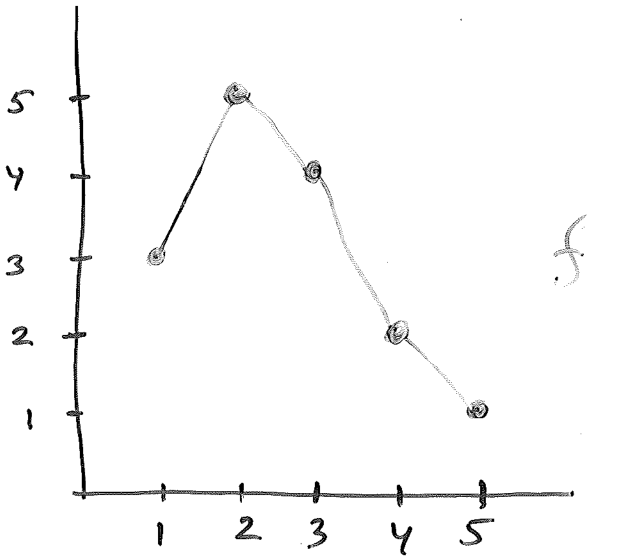
$f(3) = 4$

$f(4) = 2$

$f(2) = 5$

$f(5) = 1$

linear in-between

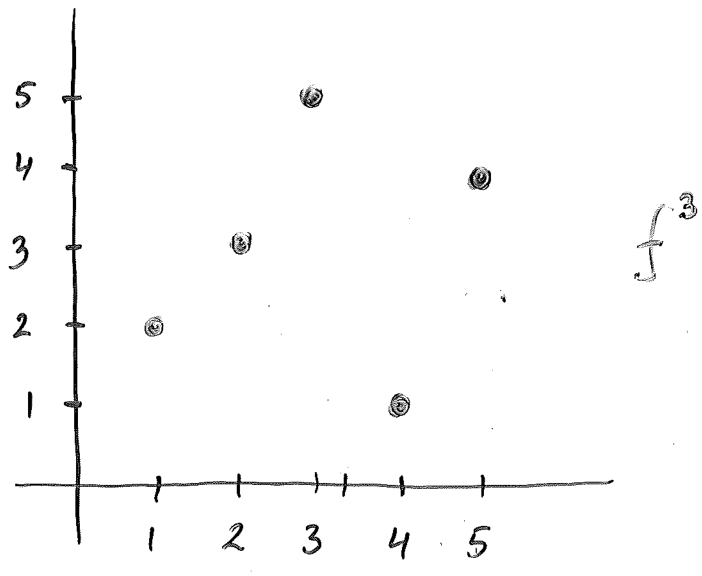


$f[1,2] = [3,5]$
$f[3,5] = [1,4]$
$f[1,4] = [2,5] = f^3[1,2]$

$f^3[2,3] = [3,5]$

$f^3[3,4] = [1,5]$ str. decr.

$f^3[4,5] = [1,4]$

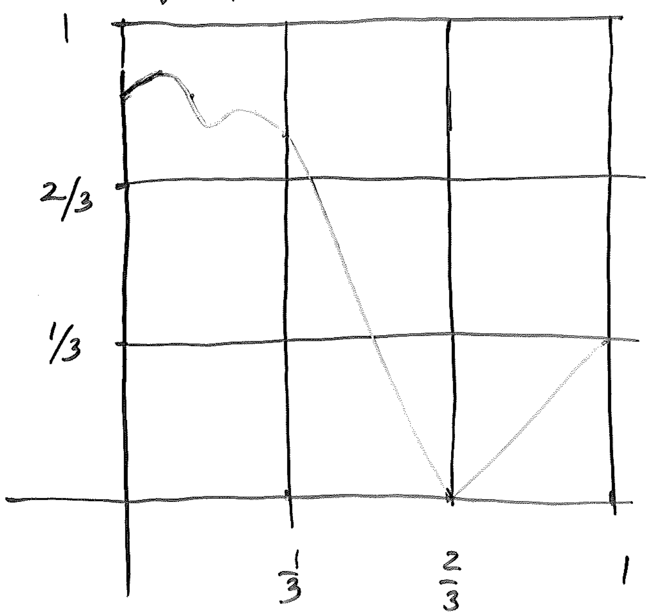


43 Similarly, we can find an f with period 7 but not 5, and so on.

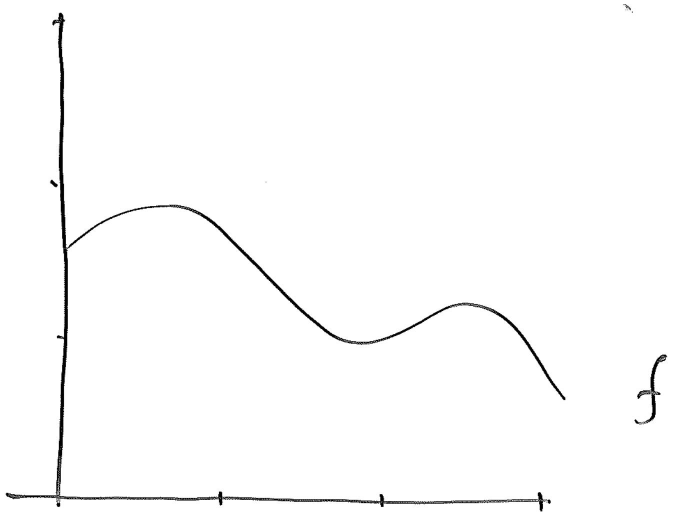
Doubling transformation

I closed bounded interval

f given, continuous. Construct F, the "double of f".



F



f

$F(x) = \frac{2}{3} + \frac{1}{3} f(3x)$ on $[0, \frac{1}{3}]$

$F(\frac{1}{3}) = \frac{2}{3} + \frac{1}{3} f(1)$ linear on $[\frac{1}{3}, \frac{2}{3}]$

$F(\frac{2}{3}) = 0$

$F(1) = \frac{1}{3}$ linear on $[\frac{2}{3}, 1]$

44 F has a fixed point in $[\frac{1}{3}, \frac{2}{3}]$, unique.
F has period $2n \iff f$ has period n

$$F : [0, \tfrac{1}{3}] \to [\tfrac{2}{3}, 1]$$
$$F : [\tfrac{2}{3}, 1] \to [0, \tfrac{1}{3}]$$

F has a fixed point $p \in [\frac{1}{3}, \frac{2}{3}]$. Repelling!
If $x \neq p$, $x \in [\frac{1}{3}, \frac{2}{3}]$, then $F^n(x) \in [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ eventually. Thus x is <u>not</u> a periodic point.

Suppose $f^n(x) = x$. Consider $F^{2n}\left(\frac{x}{3}\right)$.

Firstly, $F^2\left(\frac{x}{3}\right) = F\left(\frac{2}{3} + \frac{1}{3}f(x)\right) = \frac{1}{3}f(x)$.

Secondly, $F^4\left(\frac{x}{3}\right) = F^2\left(F^2\left(\frac{x}{3}\right)\right) = F^2\left(\frac{f(x)}{3}\right)$

$= \frac{1}{3}f(f(x)) = \frac{1}{3}f^2(x)$ and hence

$$F^{2n}\left(\frac{x}{3}\right) = \frac{1}{3}f^n(x)$$

Then if $f^n(x) = x$ we get

$$F^{2n}\left(\frac{x}{3}\right) = \frac{1}{3}f^n(x) = \frac{x}{3}.$$

$\therefore \frac{x}{3}$ periodic with per. $2n$

45 Let p be periodic for F, p not a fixed point. Then $p \in [0, \frac{1}{3}]$ or $p \in [\frac{2}{3}, 1]$.

In the latter case $F(p) \in [0, \frac{1}{3}]$.

If the period is k then k is even $= 2n$.

This is because $F^{2n}[0, \frac{1}{3}] \subset [0, \frac{1}{3}]$ and $F^{2n+1}[0, \frac{1}{3}] \subset [\frac{2}{3}, 1]$.

As above

$$F^{2n}(p) = \frac{1}{3} f^n(3p) \quad \text{in case } p \in [0, \tfrac{1}{3}].$$

Thus, if $p = F^{2n}(p) \Rightarrow 3p = f^n(3p)$.

Also

$$F^{2n}(F(p)) = \frac{1}{3} f^n(3F(p)) \quad \text{in case } p \in [\tfrac{2}{3}, 1].$$

Thus if $F^{2n}(p) = p$ we have $F^{2n}(F(p)) = F(p)$

whence

$$\begin{array}{l} 3F^{2n}(F(p)) = f^n(3F(p)) \\ \parallel \\ 3F(p) \end{array}$$

so $3F(p)$ is periodic for f.