1.9. Structural stability

<u>Def. 9.1.</u> $f, g : \mathbb{R} \to \mathbb{R}$

Call $d_0(f,g) \equiv \sup\limits_{x \in \mathbb{R}} |f(x) - g(x)|$

the C^0- distance between f and g.

and, if $f, g \in C^r$,

$$d_r(f,g) \equiv \sup_{x \in \mathbb{R}} \{ |f(x)-g(x)|, |f'(x)-g'(x)|, \ldots \\ |f^{(r)}(x) - g^{(r)}(x)| \}$$

the C^r- distance between f and g.

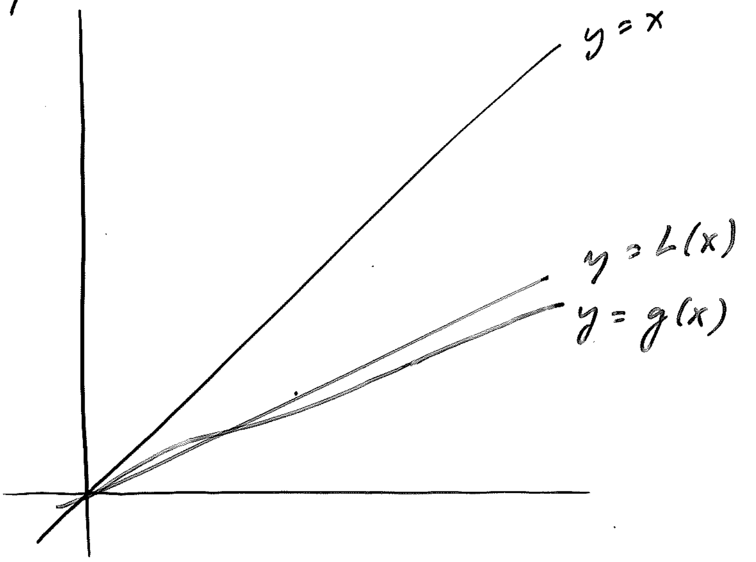
<u>Def. 9.3.</u> $f : J \to J$, $f \in C^r$.

f is <u>C^r- structurally stable</u> on J if

$\exists \varepsilon > 0 : \quad d_r(f,g) < \varepsilon \implies f \sim g$

Note: C^r- struct. stable $\Rightarrow$ C^{r+1}- struct. stable

so the minimal r is of interest.



g close to L

g' close to L'

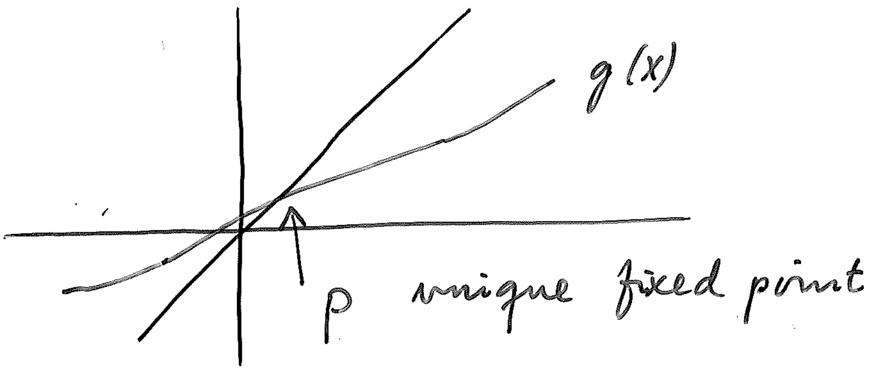
if $d_1(g, L)$ small

Ex. 9.4. $L(x) = \frac{1}{2}x$. Then L is C_1 - structurally stable, for $\varepsilon < \frac{1}{2}$.

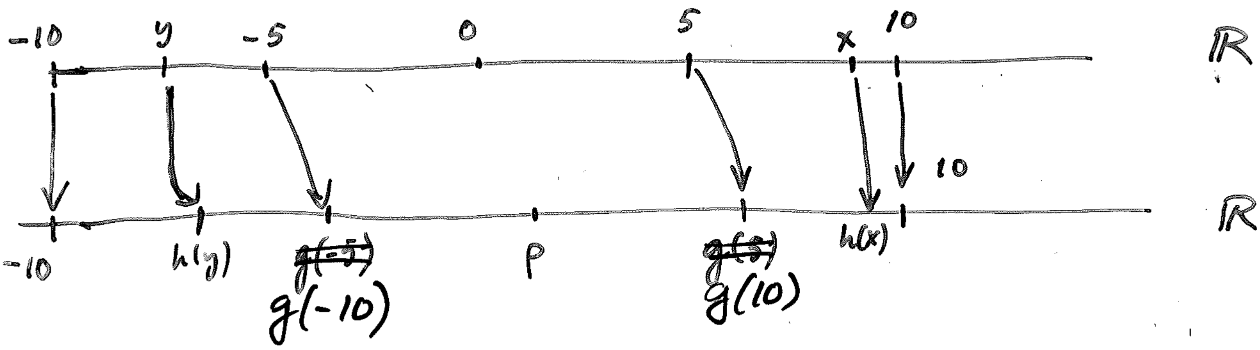
Let $|g(x) - L(x)| < \varepsilon < \frac{1}{2}$, all x

$$| g'(x) - L'(x)| = | g'(x) - \tfrac{1}{2}| < \varepsilon < \tfrac{1}{2}, \text{ all } x$$

Then $0 < g'(x) < 1$, all x $\therefore$ g contraction!



To construct the topological conjugacy h:



$$-5 > x \geq -10 \qquad \text{and} \qquad 5 < x \leq 10$$

The union of these two intervals is a fundamental domain for L. Each L-orbit (backward and forward) enters the fundamental domain exactly once (exc. for the orbit of 0, of course).

$$L^n(y) \in (5, 10] \quad \text{iff} \quad y \in (5 \cdot 2^{+n}, 10 \cdot 2^{+n}],$$

n taking both positive and negative values.

For g the intervals $(g(10), 10]$ and $[-10, g(-10))$ make up a fundamental domain.

$$g^n(y) \in (g(10), 10] \quad \text{iff} \quad y \in (g^{-n+1}(10), g^{-n}(10)]$$

Again, all orbits, except that of the fixpoint p, visit the fundamental domain exactly once.

Define $h(10) = 10$, $h(5) = g(10)$ and (for instance) linearly on $(5, 10]$. h is, of course, a homeomorphism on $(5, 10]$. We note that

$$h(L(10)) = h(5) = g(10) = g(h(10)).$$

For $x > 0$ we find an n with $L^n x \in (5, 10]$.
unique!

Then $h(L^n x) \in (g(10), 10]$ is well-defined and so is $g^{-n}(h(L^n(x)))$. Call this number $h(x)$:

$$h(x) = g^{-n}\left(h\left(L^n x\right)\right)$$

For $x<0$ we proceed similarly. Finally $h(0)$ is defined to be p, the unique fixed point of g.

By construction $h = g^{-n} \circ h \circ L^n$

$$g^n \circ h = h \circ L^n \quad \text{on } (5\cdot 2^{+n}, 10\cdot 2^{+n}]$$

Then for $L(x)$ in $(5\cdot 2^{+n}, 10\cdot 2^{+n}]$ we get

$$g^n \circ h\left(L(x)\right) = h\left(L^{n+1}(x)\right) \qquad (1)$$

For $L(x) \in (5\cdot 2^{+n}, 10\cdot 2^{+n}]$ we must have $x \in (5\cdot 2^{+n+1}, 10\cdot 2^{+n+1}]$ where

$$g^{n+1} \circ h = h \circ L^{n+1} \qquad (2)$$

(1) & (2) give

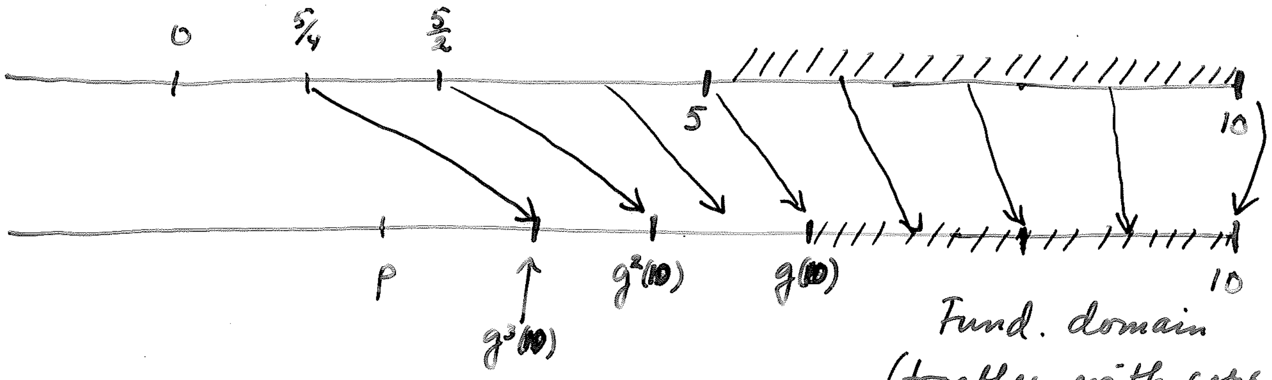
$$g^n \circ h \circ L = h \circ L^{n+1} \quad \text{on one hand}$$

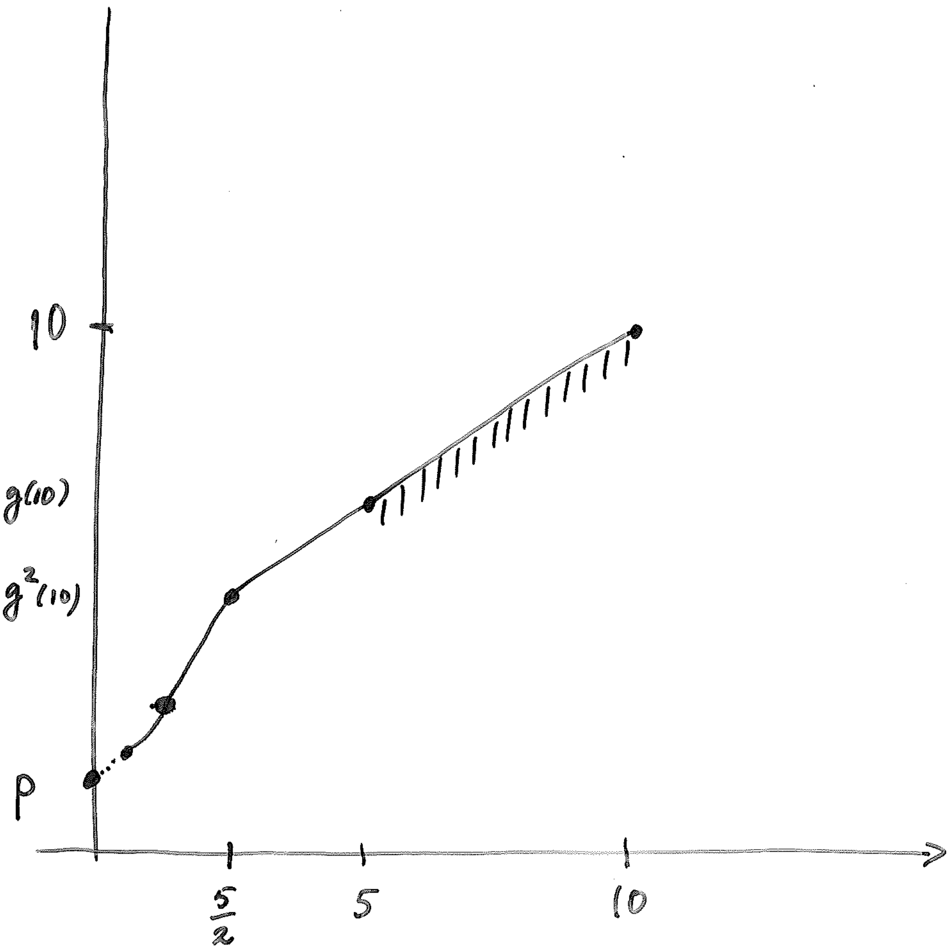
$$g^{n+1} \circ h = h \circ L^{n+1} \quad \text{on the other}$$

The 2nd eq. $\Rightarrow$ $g^n \circ g \circ h = g^n \circ h \circ L$

whence (since g^n is invertible!)

$$g \circ h = h \circ L.$$





h is monotone (strictly increasing):

$x < y$ in fund. domain then $h(x) < h(y)$

If $L^n(x) < L^n(y)$ in fund. domain then

$$h(L^n(x)) < h(L^n(y)) \quad \underline{\text{and}}$$

$$h(x) = g^{-n}(h(L^n(x)) < g^{-n}(h(L^n(y))) = h(y)$$

g str. increasing
since $g' \in (0,1)$.

If $L^n(x)$, $L^m(y)$ in fund. domain and $n > m$

then we have 1. $x > y$

$$2. \quad h(x) = g^{-n}(h(L^n(x))) \geq g^{-n}(g(10))$$

$$h(y) = g^{-m}(h(L^n(x))) \leq g^{-m}(10)$$

$$\therefore \; h(x) > g^{-(n-1)}(10) \geq g^{-m}(10) \leq h(y).$$

Hence h is strictly increasing on $(0, \infty)$.

h is continuous at 0 since

$$\lim_{x \to 0} h(x) = \lim_{n \to \infty} h(L^{+n}(10))$$

$$= \lim_{n \to \infty} g^{+n}(10) = p$$

$\therefore$ Our def. $h(0) = p$ makes h continuous.

h is clearly cont's in any $(5 \cdot 2^{-n}, 10 \cdot 2^{-n})$. What about the endpoints?

$$\lim_{\substack{x \to 5 \cdot 2^{-n} \\ x < 5 \cdot 2^{-n}}} h(x) = \lim g^{+n+1}(h(L^{-(n+1)}(x))$$

$$= g^{+n+1}(h(10)) = g^{+n+1}(10).$$

$$\lim_{\substack{x \to 5 \cdot 2^{-n} \\ x > 5 \cdot 2^{-n}}} = \lim_{\substack{x \to 5 \cdot 2^{-n} \\ x > 5 \cdot 2^{-n}}} g^{+n}(h(L^n(x)) = g^{+n}(g(10)).$$

$\therefore$ h cont's

h strictly monotone $\Longrightarrow$ h^{-1} continuous, too.
(or we can do everything for h^{-1} as for h!)

34. h is not differentiable, in general.

If it were, then

$$g'(h(x)) \cdot h'(x) = h'(L^2(x)) \cdot L'(x)$$

For $x=0$

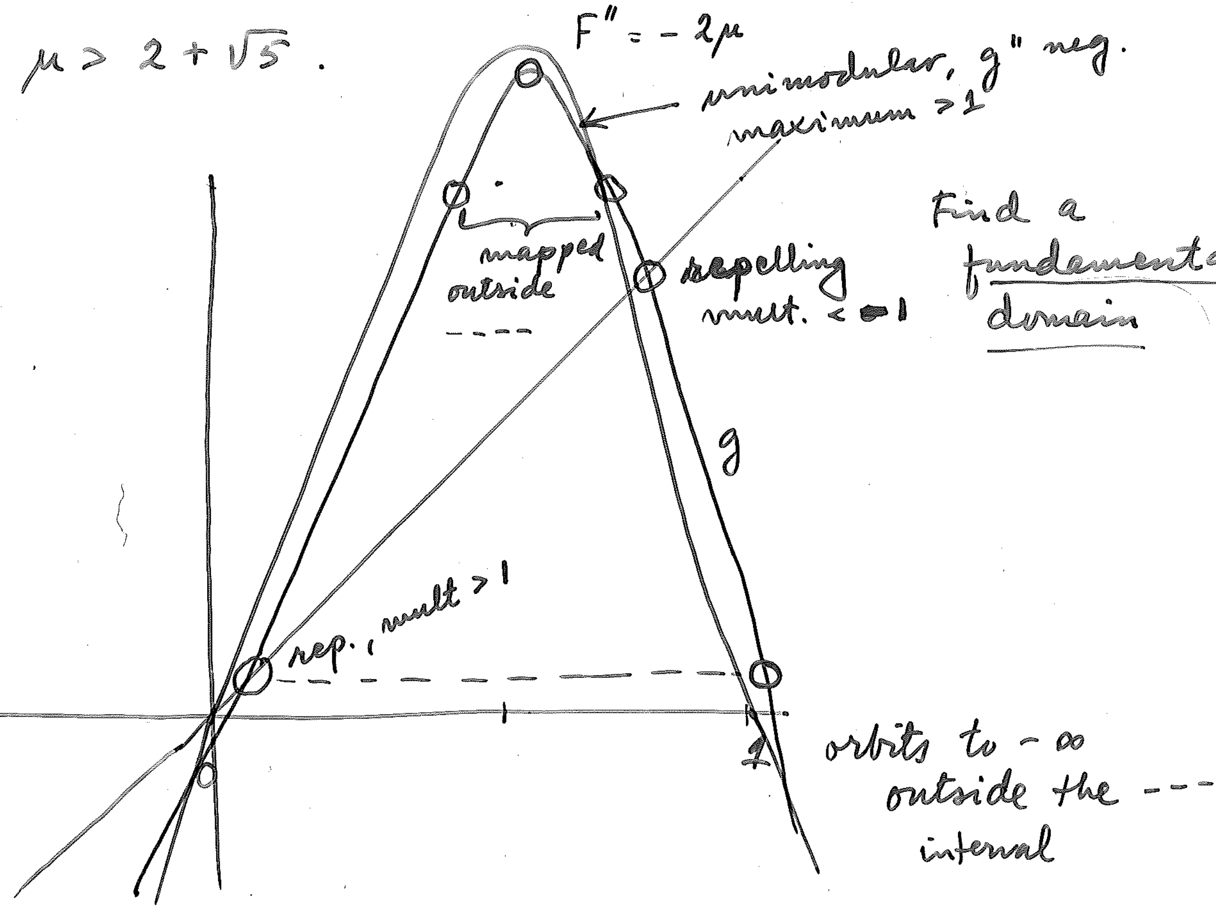
$$g'(p) \cdot h'(0) = h'(0) \cdot L'(0)$$

$$\therefore \quad g'(p) = \tfrac{1}{2}$$

We would get $g'(p)$ exactly $L'(0)$, hence no choice at all in d_1-metric. [This argument holds for nonlinear maps L as well, see p. 58]

Theorem 9.5. F_μ is structurally stable (C^2) if $\mu > 2+\sqrt{5}$.

Ideas



Find a fundamental domain

Ex. 9.6. $F_1(x) = x - x^2$ is <u>not</u> C^r struct. stable

Ex. 9.7.

$$T_\lambda(x) = x^3 - \lambda x$$

T_1 is not struct. stable

f is C^1-structurally stable <u>locally</u> at a hyperbolic fixed point.

Hartman's theorem (Th 9.8)

Let p be a hyperbolic fixed point and let $f'(p) = \lambda$, $\lambda \neq 0, 1, -1$. Then there are neighborhoods U of p and V of 0 such that

$$f|_U \sim L|_V$$

where $L(x) = \lambda x$.

Cor. $\exists\, \varepsilon > 0$ and $U \ni p$

$$d_1(f, g) < \varepsilon \implies g|_{U_0} \sim f|_U .$$

Pf. Both f, g are conjugate to L. Ideas from $g \sim \frac{1}{2}x$ above!