1 Phase portraits for equations of the type $x'' + f(x) = 0$

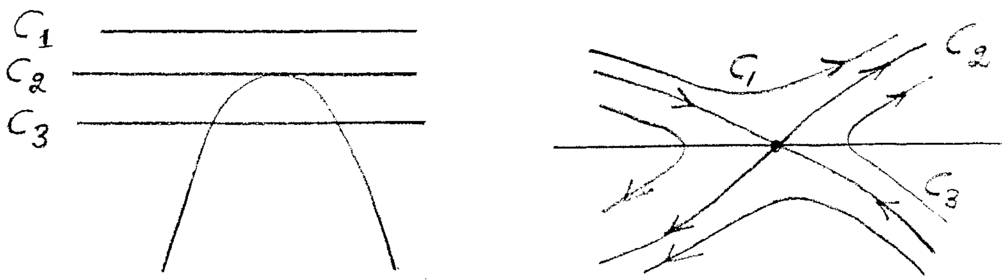
The phase portrait of the equation $x'' + f(x) = 0$ can easily be plotted if we know the graph of the integral $F(x)$ to the function $f(x)$. The equation can be written as a system $x' = y,\ y' = -f(x)$.

The solutions can be written in the form $y^2/2 + F(x) = C$, where C is a constant, implying $y = \pm\sqrt{2}\sqrt{C - F(x)}$. Indeed differentiating $y^2/2 + F(x)$ with respect to the system we get $yy' + f(x)x' = 0$ and thus it must be a constant.

The equilibria are found where $y = 0,\ f(x) = 0$, and the Jacobian is

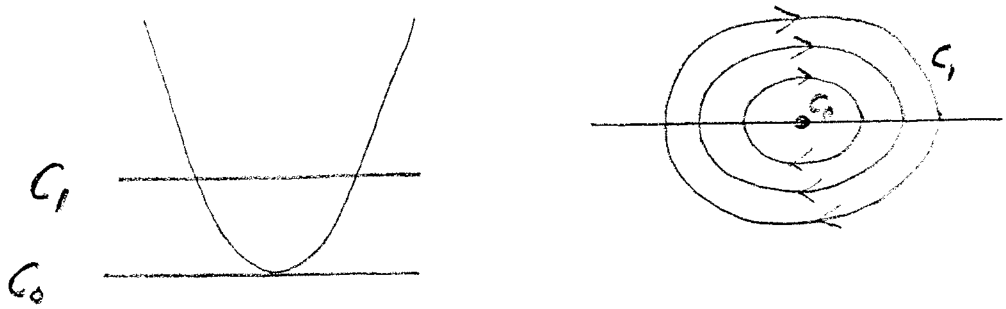
$$\begin{pmatrix} 0 & 1 \\ -f'(x) & 0 \end{pmatrix}$$

If at the equilibrium $f'(x) < 0$ the equilibrium is a saddle. This gives a maximum for $F(x)$ because at the equilibrium $f(x) = 0$. For example, if F has only one maximum and no other critical points the phase portrait looks like in the figure below.



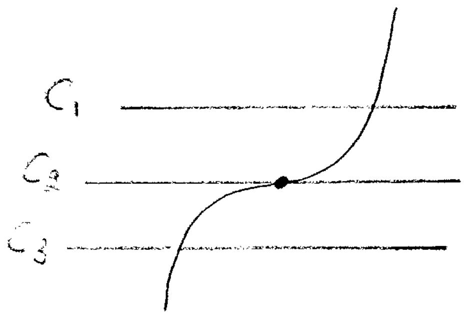
If C is the value of the maximum the solutions are the stable and unstable sets of the saddle. If C is greater than the maximum the solutions are defined for all x and if it is less than the maximum the solutions are defined for $x < x_-$ and $x > x_+$, where x_- and x_+ are the x-values where the line $x = C$ intersects the graph of F. Notice that the time direction above the x-axis is always to the right and below always to the left because of the sign of $x' = y$.

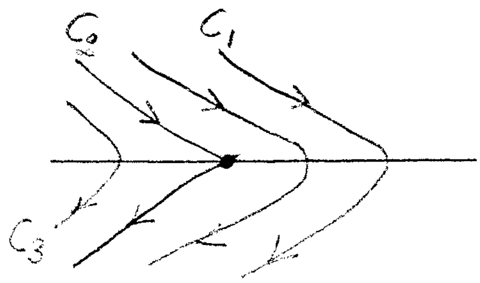
If $f'(x) > 0$ the equilibrium is a center. (Note that Grobman-Hartman theorem does not work, but symmetry gives closed curves. The parts of the curve in $y > 0$ and $y < 0$ are reflected to each other with resp to the x-axis. Thus, for example, a focus is not possible). In this case F has a minimum at the x giving the equilibrium and there are no other critical points. The phase portrait looks like in the figure below.



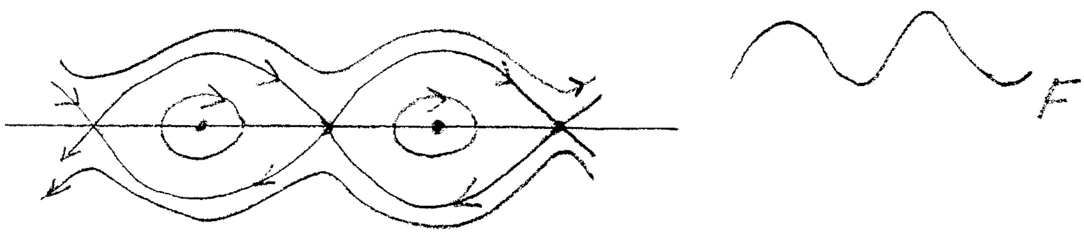
If C is a value less than the minimum we get no solution curves. If C is the minimum we get the center point and if C is greater than the minimum we get one of the closed curves.

If $f'(x) = 0$ at the equilibrium there might still be a maximum or minimum of F as above or there might be an inflection point like in the figure with the phase portrait below.

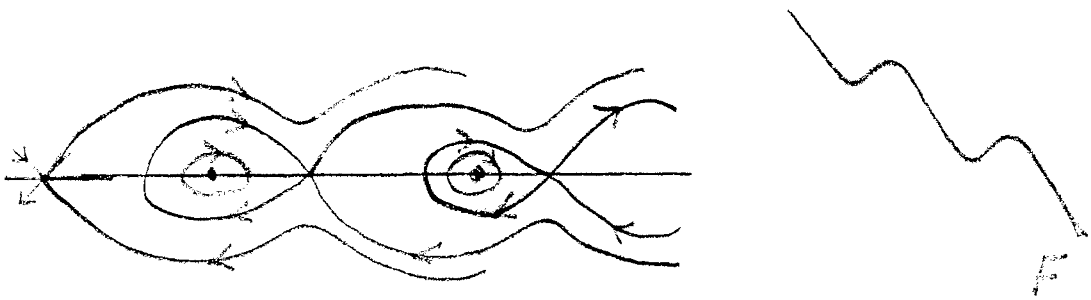




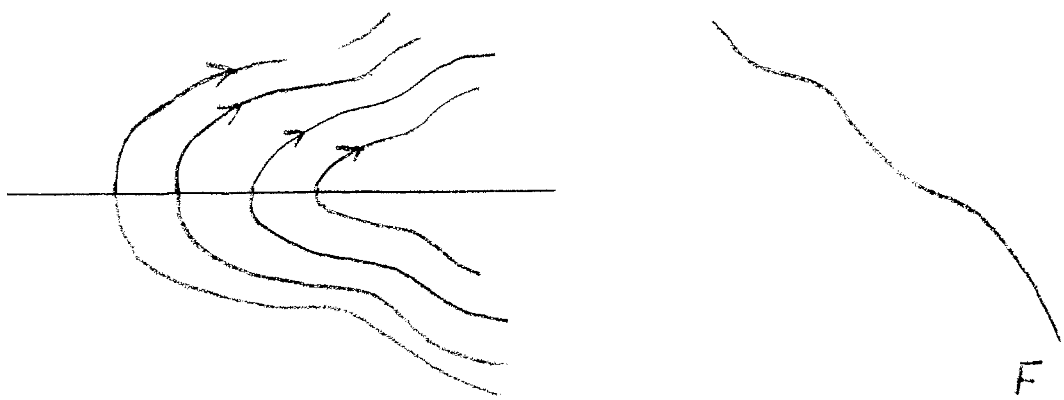
Example: One important example is the pendulum equation $x'' + \sin x = m$. For $m = 0$ the phase portrait looks like.



For $0 < m < 1$ the phase portrait looks like.



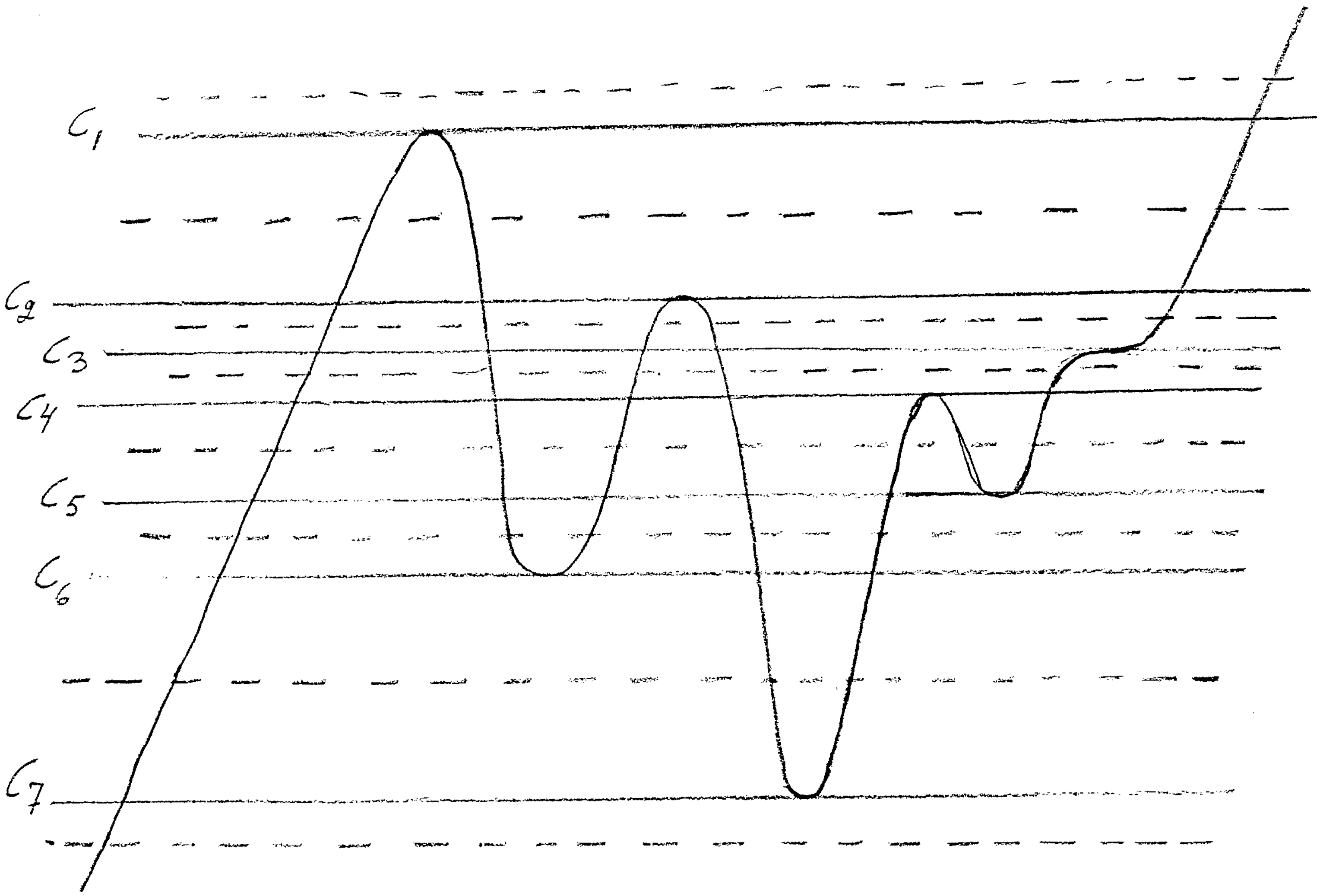
For $m > 1$ it looks like.

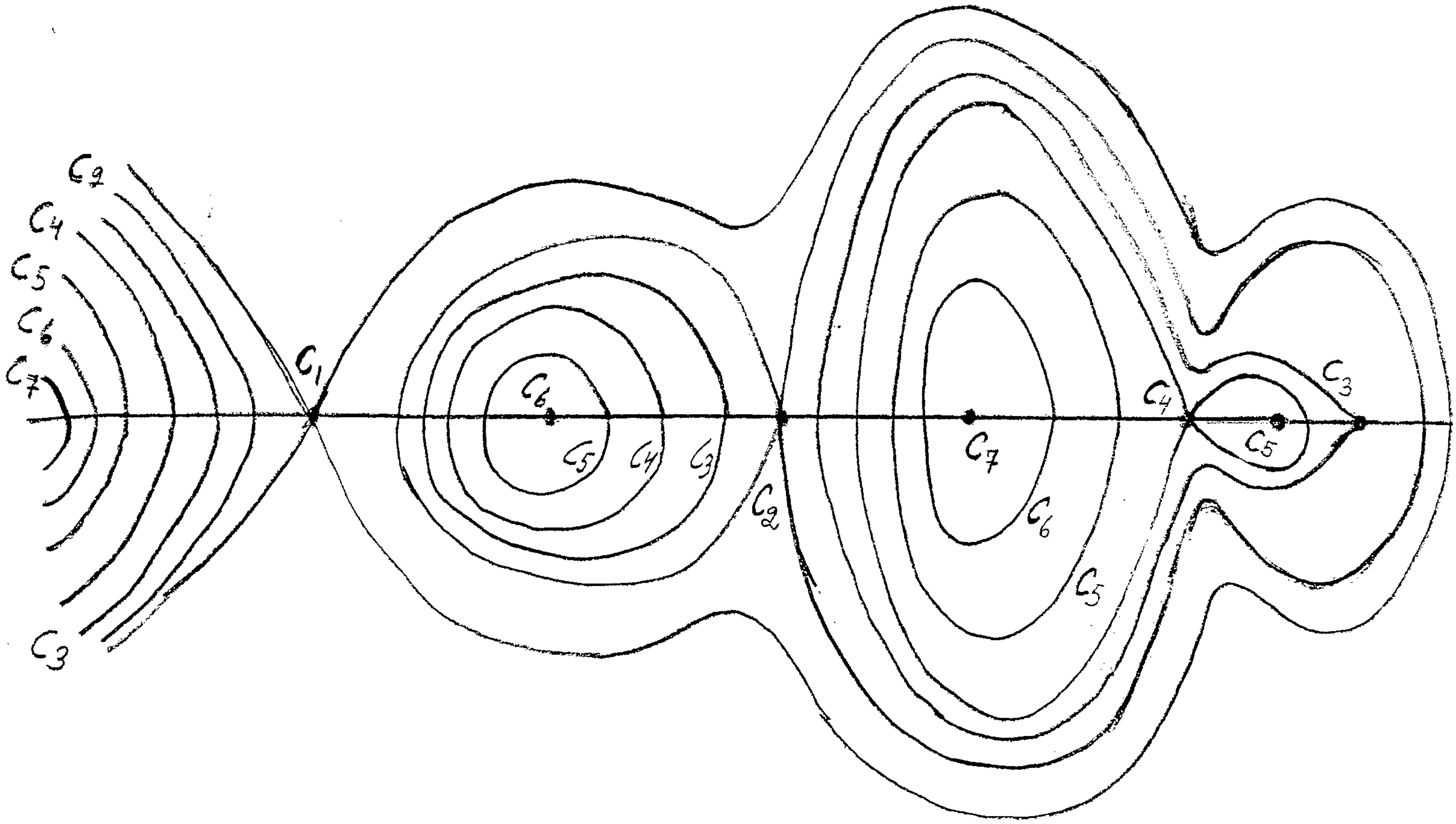


Exercise 1. Plot the phase portraits for the Duffing equation

$$x'' + ax + x^3 = 0$$

Example. A more complicated phase portrait is plotted for the graph below. First solution curves for some marked constants are plotted and only after that we give the whole phase portrait.





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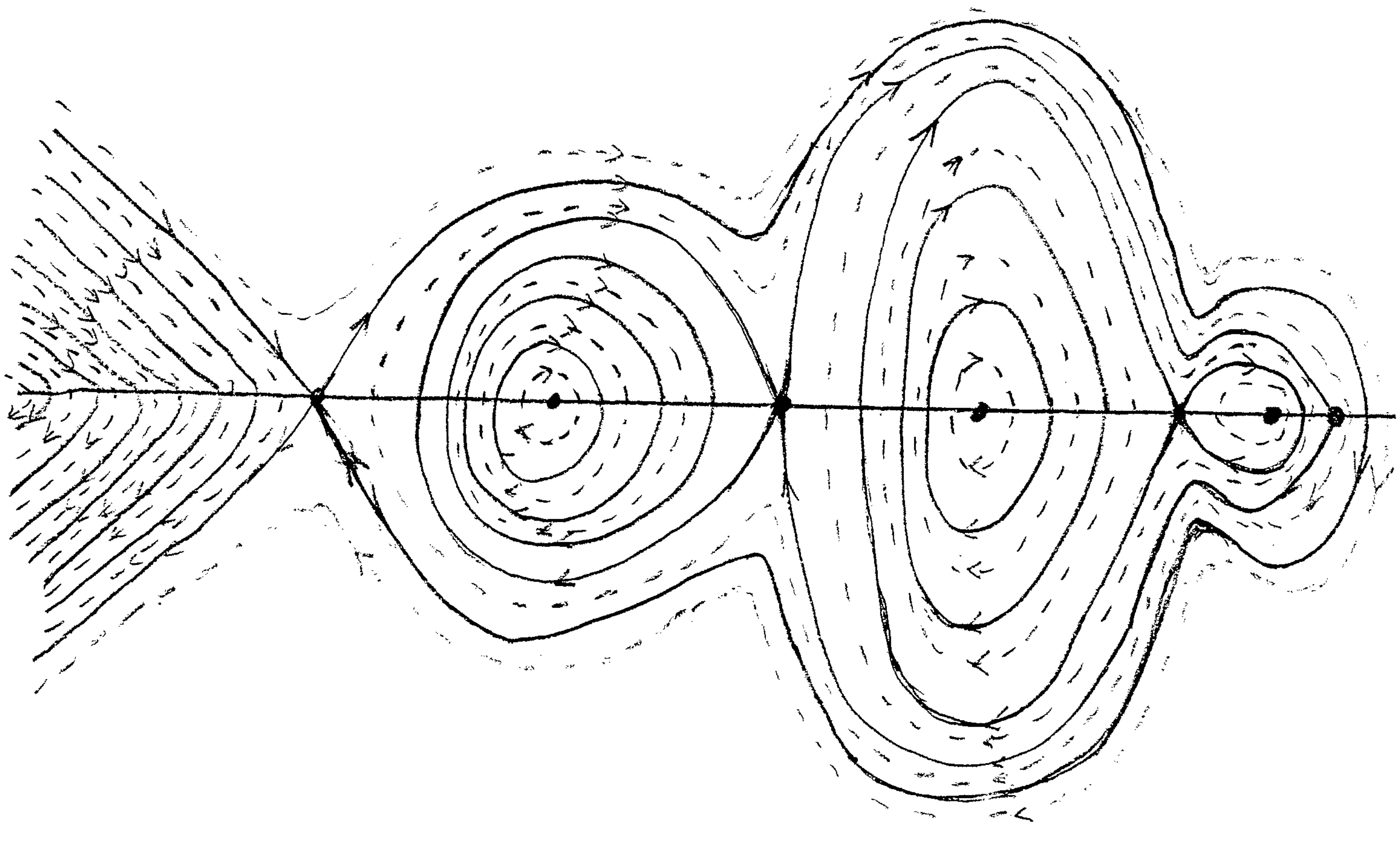

Exercise 2. Plot the phase portrait to the system where the graph of F is given in the figure below.

Exercise 3. Plot the phase portrait if f is given below

a) $f(x) = x^4 - 2x^3 - 5x^2 + 6x$

b) $f(x) = x^4 - 2.5x^3 + 0.5x^2 + x$

c) $f(x) = x^4 + 2.3x^3 - 11.2x^2 + 10.9x - 3$

Exercise 4. Plot the phase portrait for the equation $15x'' + 3x^5 - 35x^3 - 45x^2 = 0$.

Exercise 5. Plot the possible phase portraits when f is given by $f(x) = x(x^2-1)(x-a)$ for $a > 0$

Suppose $F(x) \to \infty$ for $x \to \pm\infty$ and $F(x_1) = F(x_2)$, $f(x_1) = f(x_2) = 0$ implies $x_1 = x_2$.

Build a tree: The root of the tree corresponds to the saddle given by the greatest maximum of F. If the saddle corresponding to a node a (in the tree) is in the left (right) "ögla" (the region bounded by the separatrices of the saddle) of the saddle corresponding to the node b the a lies to the left (right) of b in the tree (a descendent to b).

Ex

F(x)