

## 1 Phase portraits for one dimensional systems

We will here examine the phase portrait for one dimensional systems of differential equations of the type

$$x' = f(x),$$

where  $x \in R$ . The phase portrait consists of equilibria and trajectories between them going from one equilibrium to another. The equilibria are the solutions to  $f(x) = 0$ . If on an interval between two equilibria  $f(x) > 0$ , then the trajectory tends to the left equilibrium for  $t \rightarrow -\infty$  and to the right for  $t \rightarrow \infty$ . This happens because  $x' > 0$  means that  $x$  is increasing. Analogously if  $f(x) < 0$  on an interval between two equilibria, then the trajectory tends to the right equilibrium for  $t \rightarrow -\infty$  and to the left for  $t \rightarrow \infty$ .

If we consider the simple linear case  $x' = \lambda x$  we get the phase portrait below if  $\lambda > 0$ :

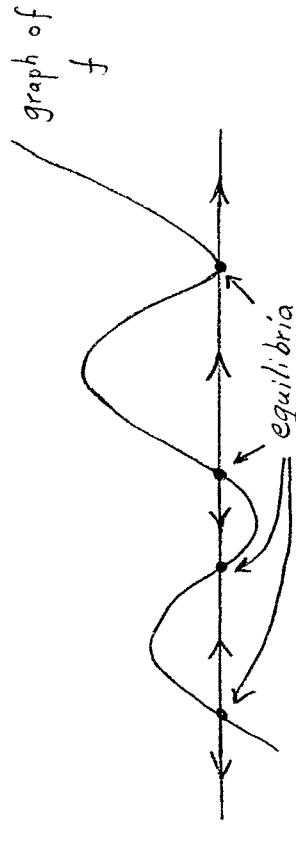


Clearly the solutions decrease for  $x < 0$  and tend to negative infinity and increase for  $x > 0$  and tend to positive infinity. Zero is an equilibrium which is unstable. If  $\lambda < 0$  we get the phase portrait



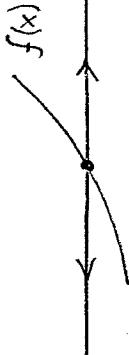
where the equilibrium zero is stable.

Knowing the graph of  $f$  is enough for plotting phase portrait. See fig below.



About the type of an equilibrium we can conclude the following:

Let  $x_0$  be an equilibrium. If near  $x_0$ ,  $f(x) > 0$  for  $x > x_0$  and  $f(x) < 0$  for  $x < x_0$  then  $x_0$  is unstable (in both directions). This means that if  $f(x_0) = 0 < f'(x_0)$ , then  $x_0$  is an unstable equilibrium.



If near  $x_0$ ,  $f(x) < 0$  for  $x > x_0$  and  $f(x) > 0$  for  $x < x_0$  then  $x_0$  is stable. This means that if  $f(x_0) = 0 > f'(x_0)$ , then  $x_0$  is a stable equilibrium.



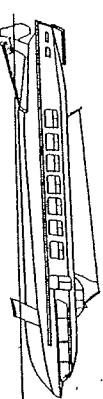
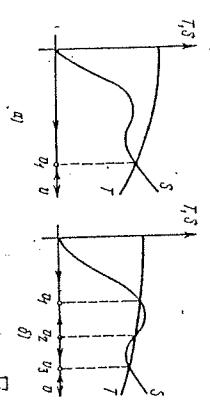
If  $f(x) < 0$  for all  $x$  near  $x_0$ ,  $x \neq x_0$  or  $f(x) > 0$  for all  $x$  near  $x_0$ ,  $x \neq x_0$  then  $x_0$  is a so called shunt. It is unstable but if we start from one side of the equilibrium it is attracting although small (in practice always existing) perturbations when we are near to the equilibrium will push it to the other side after which we will move away.



Let us look at the nonlinear equation  $x' = (x^2+1)(x-1)(x-2)(x-3)(x-4)$ . The equation is a bit hard to integrate if we do not use symbolic integration programs. However we can immediately plot the phase portrait. The equilibria are 1,2,3 and 4 and the phase portrait looks like:



As another example look at the movement of a hydrofoil described by the equation  $v' = T(v) - S(v)$ , where  $v$  is the speed of the boat,  $T$  is the force caused by the motor and  $S$  is the friction force of the water. The graphs of the functions are demonstrated in figure below. The phase portrait is on the  $v$ -axis for two different situations.



#### Bifurcation diagrams with one parameter

Let us now look at an example with parameter:  $x' = ax^2 - a^2$ . If  $a < 0$  there are no equilibria and the phase portrait looks like:

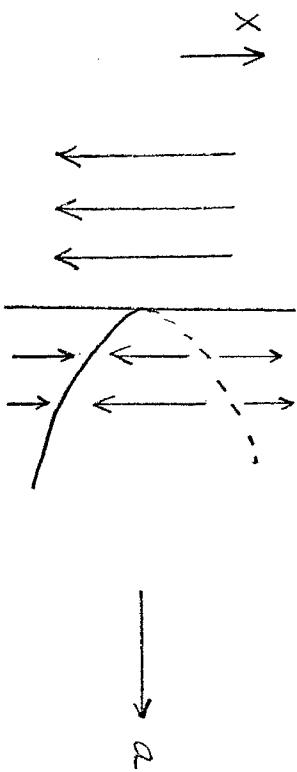


If  $a > 0$  there are two equilibria  $-\sqrt{a}$  and  $\sqrt{a}$ . The former is stable and the last one is unstable. The phase portrait looks like:



(Observe that in these cases the solution may tend to infinity for finite time). If  $a = 0$  there are infinitely many equilibria.

We can now summarize the situation in a so called bifurcation diagram below

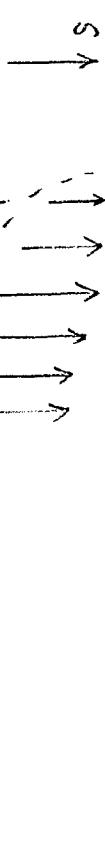


Here the horizontal axis corresponds to the parameter  $a$  and the phase portrait is along the vertical axis corresponding to the variable  $x$ . The curves represent the equilibria and the dotted curves mean that the equilibrium is unstable.

Another example of such bifurcations diagram we get from the system

$$s' = m - \frac{bs}{s^2 + a^2}, \quad m, a, b, s > 0$$

describing the slip of an asynchronous electric machine.



Let us look at yet another example with parameter:  
 $x' = ax^2 - a^2x + 2a^2x + 2ax + 3x - a^4 + a^3 - a^2 + 3a + 6$ .

The right side is a polynomial of second degree and thus we can have no more than two equilibria. The extremum of the function is of great importance for determining the phase portrait. It can be calculated to (use, for example, maple)

$$\frac{(a^3 - 2a + 3)^2}{4a}.$$

Let us first consider the cases when  $a^3 - 2a + 3 \neq 0$ . If  $a > 0$  the extremum is a minimum which is less than zero and the phase portrait looks like



If  $a < 0$  the extremum is a maximum greater than zero and the phase

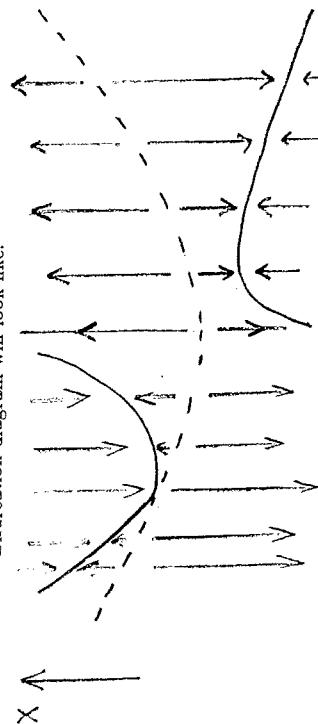
portrait looks like



For  $a$  about -1.893.. (and only then)  $a^3 + 2a + 3 = 0$  the function has a maximum equaling zero and the phase portrait looks like



Bifurcation diagram will look like:

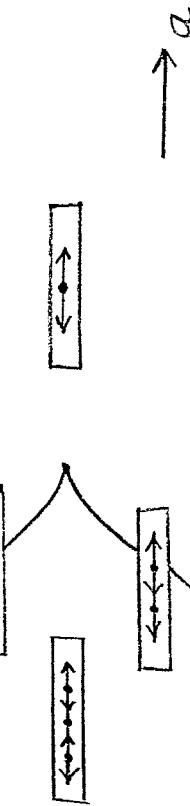


Bifurcation diagrams with two parameters

Let us now look at an example with two parameters  $x' = x^3 + ax + b$ . For  $a \geq 0$  the function in the right hand side is growing and has only one root leading to the phase portrait



$b$



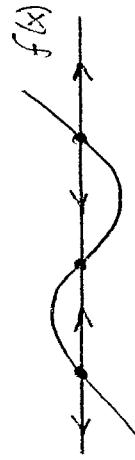
If  $a < 0$  and  $b > -\frac{2a\sqrt{-a}}{3\sqrt{3}}$  both the maximum and the minimum of the function are above zero and we again get the same phase portrait



If  $a < 0$  and  $b < -\frac{2a\sqrt{-a}}{3\sqrt{3}}$  both the maximum and the minimum of the function are below zero and we again get the same phase portrait



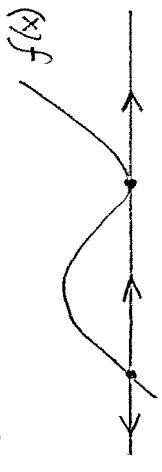
If  $a < 0$  and  $\frac{2a\sqrt{-a}}{3\sqrt{3}} < b < -\frac{2a\sqrt{-a}}{3\sqrt{3}}$  the maximum is greater than zero and the minimum is less than zero and we get the phase portrait



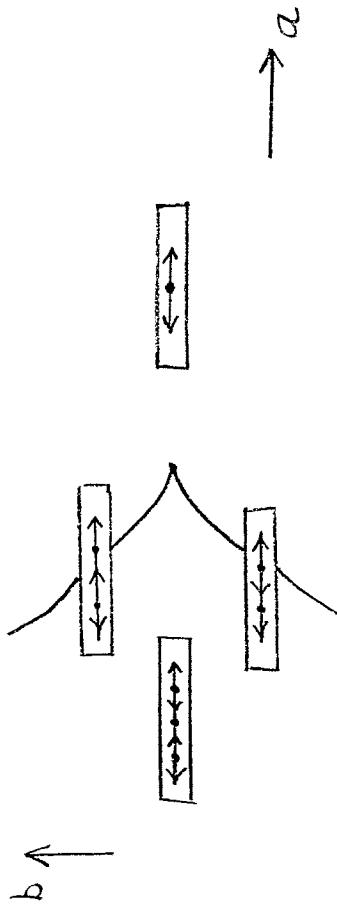
If  $b = \frac{2a\sqrt{-a}}{3\sqrt{3}}$  the maximum is zero and the minimum below and we get the phase portrait



If  $b = -\frac{2a\sqrt{-a}}{3\sqrt{3}}$  the minimum is zero and the maximum above and we get the phase portrait



The whole situation can be summarized in one bifurcation diagram where the  $a, b$ -plane is divided into pieces having equivalent phase portraits seen in so called windows.



Exercise 1.1 Construct the phase portrait for the one dimensional systems

- a)  $x' = c - x^2$
- b)  $x' = x^3 + cx$
- c)  $x' = x^4 - 5x^2 + c$
- d)  $x' = x^5 + 2cx$
- e)  $x' = 2c - x^2 - x^4$
- f)  $x' = x^4 + 6x^2 + c$
- g)  $x' = x^4 + cx^2$

- h)  $x' = x^5 + cx^3$   
 i)  $x' = cx^4 + x^2$   
 j)  $x' = xe^x + c$   
 k)  $x' = 1/(1+x^2) + c$   
 l)  $x' = cx - x^3$   
 m)  $x' = x^4 - 8x^3 + 22x^2 - 36x + c$   
 n)  $x' = 3x^4 - 16x^3 + 3x^2 + 36x + c$   
 o)  $x' = 3x^4 - 8x^3 - 15x^2 + 36x + c$   
 p)  $x' = 2x^3 - 3x^2 + 12x + c$   
 q)  $x' = 2x^3 - 3x^2 - 12x + c$

for different values of the parameter  $c$ . Summarize the results in a bifurcation diagram.

Exercise 1.2 Plot the graph of the right side for the system

$$x' = \frac{10000x^3 - x}{1 + 10000x^2}.$$

Plot the phase portrait.

Exercise 1.3 Plot the phase portrait for the system

- a)  $x' = ax^2 + (a+b)x + b$   
 b)  $x' = x^3 + (a+1)x^2 + b$   
 c)  $x' = x^3 + ax^2 + b + a^3$   
 d)  $x' = x^3 + (b-1)x^2 + a + 1$   
 e)  $x' = x^3 + bx^2 + a - b^3$   
 f)  $x' = x^4 + (a+1)x^3 + b$   
 g)  $x' = x^4 + ax^3 + b + a^3$   
 h)  $x' = x^4 + (b-1)x^3 + a + 1$   
 i)  $x' = x^4 + bx^3 + a - b^3$   
 j)  $x' = 5 - x/(ax^2 + bx + 4)$   $\rightarrow b^2 < 16a$   
 k)  $x' = 2 - bx/(ax^2 + x + a)$   $a > 1/2$   
 l)  $x' = ax^2 + bx + a + 2$   
 m)  $x' = ax^2 + (b+1)x + a$   
 n)  $x' = bx^2 + ax + b + 1$   
 o)  $x' = (a+1)x^2 + bx + a$   
 p)  $x' = ax^2 + (b-1)x + a$   
 q)  $x' = ax^2 + bx + a + b$   
 r)  $x' = ax^2 + (b+a)x + a$   
 s)  $x' = bx^2 + bx + a + 2$   
 t)  $x' = ax^2 + (b-a)x + a$   
 u)  $x' = bx^2 + (a-b)x + b$

for different parameter regions for the parameters  $a$  and  $b$ . Summarize the results in a bifurcation diagram with windows.

Exercise 1.4 Find the possible different phase portraits for  $x' = 2x^5 - 10ax^3 + 8a^4x + 2a - 1$  when  $0 \leq a \leq 1$ . Hint: Use Maple to animate the curve and find that there are at least three different phase portraits for  $a = 0, 0.45, 1$ .

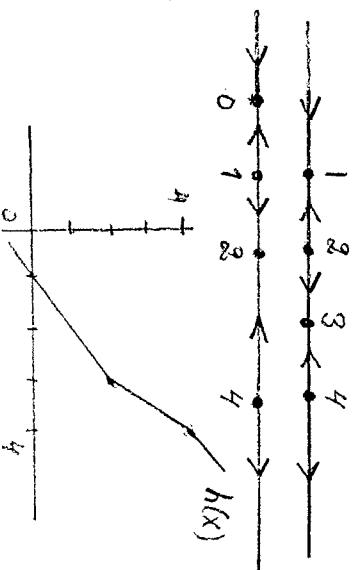
Exercise 1.5 Find the possible phase portraits in the previous exercise when  $-2 \leq a \leq 0$ .

Two one dimensional systems are topologically equivalent iff they have the same number of equilibria and the types of the equilibria coincide in the same order.

Example,  $x' = (x^2 + 1)(x - 1)(x - 2)(x - 3)$  and  $x' = x(x - 1)(x - 2)(x - 4)$  are topologically equivalent and both have the same phase diagram shown earlier above.

A homeomorphism giving the equivalence can be

$$h(x) = \begin{cases} x - 1 & x \leq 3 \\ 2x - 4 & 3 \leq x \leq 4 \\ x & x \geq 4 \end{cases}$$



Exercise 1.6 Show that the phase portrait of the system  $x' = (x+1)(x-1)(x-3)(x-4)$  is equivalent to the systems above. Find a homeomorphism giving the equivalence between one of the systems above and this system.