

1 Phase portraits for linear systems

Consider a two dimensional system of the type

$$(1.1) \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Clearly the system is linear. If both eigenvalues of the matrix

$$(1.2) \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

are real and different from each other You know from Your courses in linear algebra that there is a linear transform converting the system into the form

$$(1.2) \quad \begin{aligned} x' &= \lambda x \\ y' &= \mu y \end{aligned}$$

System (1.2) is easy to solve. The solutions are $x(t) = x(0)e^{\lambda t}$ and $y(t) = y(0)e^{\mu t}$. If we know λ and μ the solution curves are easily plotted. A collection of solution curves with indicated time direction demonstrating the geometrical behaviour of the system is called *phase portrait*. If for example $\mu > \lambda > 0$ the phase portrait of the system (1.2) is illustrated up to the left in figure 1. The origin is here unstable and is called an *unstable node*. It is also called a *source*. Let us look how we plot the solutions in this case. If we start on the x -axis clearly $y(0) = 0$ and we always remain on the x -axis and if we have not started in the origin we leave the axis to the infinity (because $\mu > 0$). If we start in the origin we clearly move nowhere. The origin is called an *equilibrium*. Analogously if we start on the y -axis outside origin we move along the axis to the infinity. If we have started on the positive side we go up to the positive infinity and if we have started on the negative side we go down to the negative infinity. Suppose now $x(0), y(0) > 0$. Then both the x - and the y -coordinates grow, but (because $\mu > \lambda$) the y -coordinate grows faster resulting in the curve in the first quadrant. Solutions in the other quadrants are plotted analogously. If $\lambda < \mu < 0$ the phase portrait can be seen up to the right in figure 1. The origin is now stable and is called a *stable node*. It is also called a *sink*. If $\lambda < 0 < \mu$ the phase portrait is down in figure 1 and the origin is called a *saddle*. Let us look how to plot the solutions in this case. If we start on the x -axis the y -coordinate remains zero and the solution goes along the x -axis to origin. If we reverse time the

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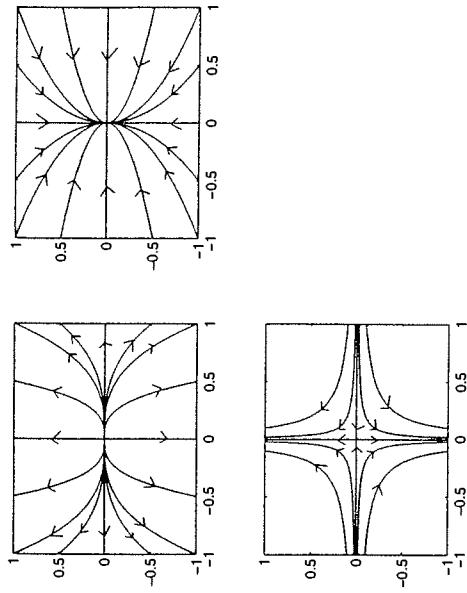


Figure 1:

System (1.2) is easy to solve. The solutions are $x(t) = x(0)e^{\lambda t}$ and $y(t) = y(0)e^{\mu t}$. If we know λ and μ the solution curves are easily plotted. A collection of solution curves with indicated time direction demonstrating the geometrical behaviour of the system is called *phase portrait*. If for example $\mu > \lambda > 0$ the phase portrait of the system (1.2) is illustrated up to the left in figure 1. The origin is here unstable and is called an *unstable node*. It is also called a *source*. Let us look how we plot the solutions in this case. If we start on the x -axis clearly $y(0) = 0$ and we always remain on the x -axis and if we have not started in the origin we leave the axis to the infinity (because $\mu > 0$). If we start in the origin we clearly move nowhere. The origin is called an *equilibrium*. Analogously if we start on the y -axis outside origin we move along the axis to the infinity. If we have started on the positive side we go up to the positive infinity and if we have started on the negative side we go down to the negative infinity. Suppose now $x(0), y(0) > 0$. Then both the x - and the y -coordinates grow, but (because $\mu > \lambda$) the y -coordinate grows faster resulting in the curve in the first quadrant. Solutions in the other quadrants are plotted analogously. If $\lambda < \mu < 0$ the phase portrait is down in figure 1 and the origin is called a *saddle*. Let us look how to plot the solutions in this case. If we start on the x -axis the y -coordinate remains zero and the solution goes along the x -axis to origin. If we reverse time the

solutions go along the axis to the infinity. Analogously a solution starting on the y -axis has come from the origin and goes along the axis to the infinity. If $x(0), y(0) > 0$ the x -coordinate decreases and the y -coordinate increases. The solution moves to the y -axis near which it goes to infinity. Solution in the other quadrants behave in an analogous manner.

Exercise 1.1 Plot the phase portrait for the solutions (1.2) in the case a)
 $\mu < 0 < \lambda$, b) $\lambda > \mu > 0$, c) $\lambda = \mu < 0$.

If the eigenvalues of the matrix A are complex the system can after a linear change of coordinates be rewritten in the form

$$(1.3) \quad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

The solutions of this systems are known from the theory of linear systems. They are $x(t) = e^{\alpha t} \cos(\beta t)$, $y(t) = -e^{\alpha t} \sin(\beta t)$. If $\alpha > 0$ and $\beta < 0$ the phase portrait of the system (1.3) is illustrated up to the left in figure 2. The origin is then called an *unstable focus*. It is also called a *source*. If $\alpha < 0$ and $\beta > 0$ the phase portrait of the system can be seen up to the right in figure 2. The origin is then called a *stable focus* or a *sink*. If $\alpha = 0 < \beta$ the phase portrait is seen down in figure 2, and the origin is called a *center*.

Exercise 1.2 Plot the phase portrait for the system (1.3) in the case a)
 $\alpha < 0 < \beta$, b) $\alpha = 0 > \beta$.

If the matrix A has a real eigenvalue λ of multiplicity 2 the system (1.1) can be transformed into the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

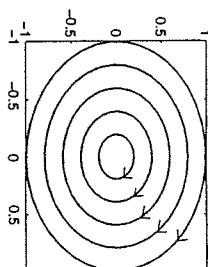
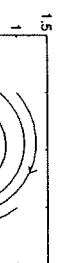
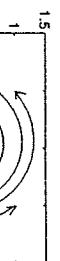


Figure 2:

The phase portrait is plotted analogously. In general the origin is called a sink for the system (1.1) if the real part of the eigenvalues of the matrix A are negative and thereby a stable node if the eigenvalues are real and a stable focus if they are complex. If the real parts of the eigenvalues are positive the origin is called a source and an unstable node if the eigenvalues are real and an unstable focus if they are complex. If the eigenvalues of the matrix A are real and of different signs the origin is called a saddle. This terminology is the same if we instead of the two dimensional system take a linear system of higher dimension.

Exercise 1.3 Which system of the list 1 may have their phase portraits corresponding to a given letter in figures 3-6.

Consider now a three dimensional system of the type

$$(1.4) \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

where A is a 3×3 matrix. If the eigenvalues of A are all real and positive the origin is called a source and an unstable node if the eigenvalues are real and of different signs the origin is called a saddle. This terminology is the same if we instead of the two dimensional system take a linear system of higher dimension.

Exercise 1.4 If the matrix A has one eigenvalue of multiplicity two, what is the normal form the equations (1.4) can be transformed into.

Exercise 1.5 Consider a fourdimensional system of the form

$$(1.8) \quad \begin{pmatrix} x' \\ y' \\ z' \\ u' \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix},$$

where A is a 4×4 matrix. If the matrix A has

- a) two pairs of different from each other complex eigenvalues,
- b) two real different and two complex eigenvalues,
- c) four real and different eigenvalues,
- d) one pair of complex eigenvalues of multiplicity 2,
- e) two real eigenvalues of multiplicity 2,

what is then the normal form into which the system can be transformed by a linear change of coordinates.

If all eigenvalues of the matrix A for the system (1.4) have negative real parts the origin attracts all solutions (is a global attractor), that is all solutions tend to the origin for infinite time. If the real parts are negative the origin is a repeller. All solutions tend to the origin when time tends to minus infinity and the origin is a source. The attractor origin was a sink. If the matrix A has two eigenvalues with negative real parts and one positive eigenvalue the solutions in a two dimensional eigenspace are attracted to the origin when time tends to plus infinity and the solutions in a one dimensional eigenspace (a line) are attracted to the origin when time tends to infinity. All other solutions pass by. They decrease along directions of the two dimensional eigenspace and increase along the one dimensional eigenspace. If the system is written in the normal form these eigenspaces are a coordinate plane and a coordinate axis, but in general they are other planes and lines. If the matrix A has two eigenvalues with positive real parts and one eigenvalue is negative the situation is analogous (You get the same figure by reversing time). In these cases the origin were a saddle.

The phase portraits can be plotted in a manner analogous to the two dimensional case.

Exercise 1.6 Which systems of list 2 may have their phase portraits corresponding to a given letter in figures C.

Exercise 1.7 In which case is the origin a saddle, sink or source for the systems in list 2.

Exercise 1.8 Which systems of list 3 may have their phase portraits corresponding to a given letter in figures D.

If there is an eigenvalue λ of multiplicity three there is a linear change of coordinates transforming the system into

$$(1.7) \quad \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

These are known facts from linear algebra.

Higher dimensional systems can be analyzed in the same way and the origin can be either a sink, source or saddle if the real parts of the eigenvalues of the matrix are non-zero.

When the system is transformed by a linear change of coordinates into the diagonalized normal form the coordinates are chosen in the directions of eigenvectors and eigenspaces. Thus, for example, in the two dimensional case the axis corresponding to the x - and y -axis of the system (1.2) are the lines corresponding to the directions of the eigenvectors of the system (1.1).

Look for example at the system 1) in list 4. The eigenvalue -4 corresponds to the vector $(1, 6)$ and the eigenvalue -6 to the eigenvector $(0, 1)$. They are both negative so the origin is a stable node and sink. The solution move straight in towards the origin along the lines corresponding to the directions of the eigenvectors. Other solutions also approach the origin but slower in the direction $(1, 6)$ and thus the solution curves become "more tight" there. In general if v_i are the eigenvectors corresponding to, for example, different real eigenvalues λ_i of the matrix A then the solutions are given by $\bar{x}(t) = c_1 v_1 e^{\lambda_1 t} + \dots + c_n v_n e^{\lambda_n t}$, where the c_i 's are determined by the column vector $B\bar{x}(0)$ and B is the inverse of the matrix having v_i as its column vector number i . If, for example, in the two dimensional case there are two complex eigenvalues $\alpha \pm \beta i$ and v and w are the real and complex parts of the eigenvectors then the solutions are given by

$$((c_1 v + c_2 w) \cos(\beta t) + (c_2 v - c_1 w) \sin(\beta t)) e^{\alpha t}$$

where

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = B \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$$

and B is the inverse of the matrix having v as its first column vector and w as its second.

Exercise 1.9 Which systems of list 4 may have their phase portraits corresponding to a given letter in figures E. (x-direction is horizontal and y-direction vertical)?

The equilibrium

Let us now look at a general autonomous system of differential equations of type

$$(1.9) \quad x'_1 = f_1(x_1, \dots, x_n), x'_2 = f_2(x_1, \dots, x_n), \dots, x'_n = f_n(x_1, \dots, x_n).$$

Points, where all derivatives in a system of differential equations are zero, are called equilibrium. A solution with initial conditions at an equilibrium point remains constant and goes nowhere. To solve for the equilibria we have to solve the systems we get equaling the right hand sides to zero:

$$f_1(x_1, \dots, x_n) = f_2(x_1, \dots, x_n) = \dots = f_n(x_1, \dots, x_n) = 0.$$

This is usually not easy to do in the general nonlinear case. Often we have to use numerical methods. Sometimes the problem can be really hard and we are not sure about the number of equilibrium points. The exercises below give You an idea about how to solve for equilibria.

Exercises

Find the equilibria for the following system of differential equations:

$$\text{Exercise 1.10} \quad x' = y, y' = -x + y + y^2/4 + x^2/4.$$

Find the equilibria in the region $x, y \geq 0$ for the following systems:

$$\text{Exercise 1.11} \quad x' = x(2 - x - y), y' = y(6 - 2x - y),$$

$$\text{Exercise 1.12} \quad x' = x(2 - x - y), y' = y(3 - 2x - y),$$

$$\text{Exercise 1.13} \quad x' = Ay(1 - x) - r_1 x, y' = Bx(1 - y) - r_2 y,$$

where A, B, r_1, r_2 are positive parameters. (of course, there are different equilibria for different values on the parameters)

Find the equilibria in the region $x, s \geq 0$ for the following system:

$$\text{Exercise 1.14} \quad x' = (s - \lambda)x, s' = ((1 - s)(s + a) - x)s,$$

where λ, a are positive parameters.

Find all the equilibria for the following systems

$$\text{Exercise 1.15} \quad x' = y, y' = -x + \alpha y + \beta xy + \gamma y^2 + \delta x^2,$$

$$\text{Exercise 1.16} \quad x' = x^3 - y^3 + xy^2 - yx^2 - x + y, y' = y + \lambda,$$

where $\alpha, \beta, \gamma, \delta, \lambda$ and $\lambda < 0$.

Find all the equilibria for the following system:

$$\text{Exercise 1.17} \quad x' = \sigma(y - x), y' = \varrho x - y - xz, z' = -\beta z + xy,$$

where the parameters σ, ϱ and β are positive.

Find the equilibria in the region $x, y, s \geq 0$ for the following system:

$$\text{Exercise 1.18} \quad \begin{aligned} x' &= \frac{-\lambda_1}{s+a_1} x \\ y' &= \frac{s+\alpha_2}{s+\alpha_2} y \\ s' &= (1-s-x/(s+a_1) - y/(s+a_2))s, \end{aligned}$$

where $\lambda_1, \lambda_2, a_1, a_2$ are positive parameters.

Find the equilibria in the region $x, y \geq 0$ for the following system:

Exercise 1.19

$$x' = (y - (x - 2)(x - 1))x, y' = (4x - (y - 3)(y - 1))y$$

One of them must be find numerically.

Find all the equilibria of the following systems using computer algebra and numerical methods. (They can be solved, for example, with MAPLE).

Exercise 1.20

$$x' = e^x - y, y' = x - y + \lambda, \lambda = 2, 0.5,$$

Exercise 1.21

$$x' = x - 2x^2 + y + 1, y' = 3x - y^2 + 2 + 2y,$$

Exercise 1.22

$$x' = y - x^3 - 4x^2 + 2x, y' = 1 - x - y,$$

Exercise 1.23

$$x' = -x^2 e^{-1/y} + 1 - x, y' = 2x^2 e^{-1/y} - 2(y - 1).$$

Find the equilibria of the following system in the region $\varrho \geq 1, -\pi \leq \varphi \leq \pi$ (analytically)

Exercise 1.24

$$d\varphi/dt = \varrho - \cos \varphi, d\varrho/dt = 2\varrho(\lambda - \mu\varrho - \sin \varphi),$$

where $\mu \geq 0, \lambda \geq 1$.

Local phase portrait around equilibria

According to a theorem of Groisman-Hartman, if the real parts of the Jacobian matrix at an equilibrium are non-zero, there is a change of variables in a neighbourhood of the equilibrium such that the system in the new variables becomes the linear system determined by the Jacobian matrix. The equilibrium is called a sink if the linearized system is a sink. Analogously we define sources, saddles, nodes and foci. In this case it is possible to plot the local phase portrait around the equilibrium. Let us look how this is done on one example. We take the system in the exercise 1.11. The Jacobian for the system is

$$J = \begin{pmatrix} 2 - 2x - y & -x \\ -2y & 6 - 2x - 2y \end{pmatrix}.$$

The equilibria where $(0,0)$, $(2,0)$ and $(0,6)$. At the origin it is

$$J = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$$

and clearly the eigenvalues are both positive and we have an unstable node. The eigenvectors are $(0,1)$ and $(1,0)$. The local phase portrait around the origin is shown in figure F1. The Jacobian at the second equilibrium $x = 0, y = 6$ is

$$J = \begin{pmatrix} -4 & 0 \\ -12 & -6 \end{pmatrix}.$$

It has one eigenvalue -4 with eigenvector $(1,0)$ and another eigenvalue -6 with eigenvector $(0,1)$. Because one eigenvalue is less than zero it is a stable node. The phase portrait around the equilibrium is shown in figure F2. The Jacobian at the the third equilibrium $x = 2, y = 0$ is

$$J = \begin{pmatrix} -2 & -2 \\ 0 & 2 \end{pmatrix}.$$

It has one eigenvalue -2 with eigenvector $(1,0)$ and another eigenvalue 2 with eigenvector $(-1,2)$. Because one eigenvalue is less than zero and the other greater it is a saddle. The phase portrait around the equilibrium is shown in figure F3.

Now we can plot a figure where the phase portrait is given around all the three equilibria in the region $x, y \geq 0$. This is seen in figure F4.

Now try to do it by Yourselves.

Exercise 1.25 Plot the local phase portraits for the equilibria

- a) in the exercises 1.10 and 1.12,
- b) in the exercise 1.14 for $\lambda = 0.8, a = 0.2$ and $\lambda = 2, a = 0.2$,
- c) in exercise 1.19.

Type of equilibrium

It is not always necessary to know the eigenvectors or even the exact eigenvalues for an equilibrium. The type of the equilibrium often gives enough information. Some types are determined by the signs of the real parts of the eigenvalues. As You know from linear algebra in the two dimensional case we can see this directly from the sign of the determinant and the trace. If the determinant is negative the product of the eigenvalues is negative meaning that the real parts of the eigenvalues are of different signs and thus we have a saddle. If the determinant is greater than zero we have a sink if the trace is negative and a source if the trace is positive. If p is the trace and q the determinant we have a focus when $q > p^2/4$. For example in exercise 1.11 the point $(2,0)$ gives a negative determinant ($=-4$) and the point is a saddle. The point $(0,6)$ gives a positive determinant ($=24$) and a negative trace ($=-10$) and it is a sink and stable node. Of course, in these easy cases the eigenvalues can be seen directly from the elements in the matrix, but this is not too usual as can be seen in the following exercises.

Exercise 1.26 Determine the types of the equilibria in the exercises 1.13 - 1.16.

Exercise 1.27 Use numerical methods (for example, MAPLE) to determine the type of the equilibria in exercises 1.21 - 1.23.

Exercise 1.28 Determine the type of the equilibrium in the exercise 1.20 for all λ .

Exercise 1.29 Determine the type of the origin in exercise 1.17 for different parameter regions.

```
hold off,
xmin=input('minimal x coord in plot ');
xmax=input('maximal x coord in plot ');
ymin=input('minimal y coord in plot ');
ymax=input('maximal y coord in plot ');
```

Using programs to plot phase portraits

The phase portrait can also be plotted by different programs, for example maple or matlab. Anyhow these are slower than if You do it directly by integrating procedures in a programming language.

Below You can see how the figure 1 can be done by matlab:
We use a file:

```
t0=0;t1=50; subplot(221);hold off,t=[];x=[];
xx0=[1 0 1 0 -1 1 -1 0.5 1 -0.5 1 -1 0.5 -1];
yy0=[1 1 0 -1 0 1 -1 1 0.5 1 -0.5 0.5 -1 -1 -0.5];
for i=1:16,
x0(i)=xx0(i);x0(2)=yy0(i);[t,x]=ode23('dy1',t0,t1,x0);
plot(x(:,1),x(:,2));hold on
end

subplot(222);t=[];x=[];hold off
xx0=[1 0 1 0 -1 -1 1 -1 0.5 1 -0.5 1 -1 0.5 -1];
yy0=[1 1 0 -1 0 1 -1 1 0.5 1 -0.5 0.5 -1 -1 -0.5];
for i=1:16,
x0(i)=xx0(i);x0(2)=yy0(i);[t,x]=ode23('dy2',t0,t1,x0);
plot(x(:,1),x(:,2));hold on
end

subplot(223);t=[];x=[];t1=10;hold off,
xx0=[0 0 1 -1 -1 -1 -1 1 1 1 1 ];
yy0=[0.001 -0.001 0 0 0.02 -0.02 -0.1 0.1 0.1 -0.1 -0.02 0.02];
for i=1:12,
x0(i)=xx0(i);x0(2)=yy0(i);[t,x]=ode23('dy3',t0,t1,x0);
plot(x(:,1),x(:,2));hold on
end
axis([-1 1 -1 1]);
```

The functions to be integrated are found in the files dy1.m, dy2.m and dy3.m.

```
dy1.m: function xx1 = dysns1(t,x) xx1(1) = -x(1);xx1(2) = -3*x(2);
dy2.m: function xx1 = dysns1(t,x) xx1(1) = -2*x(1);xx1(2) = -x(2);
dy3.m: function xx1 = dysns1(t,x) xx1(1) = -x(1);xx1(2) = x(2);
```

When You start with a new phase portrait You are usually not sure where to put them and then the following general program can be of use in the two dimensional case. Here You can choose where to start a trajectory by using the mouse and determining the lenght in time.

```
% program for plotting 2 dim phase portraits using function in fund2.m
% the parameters in the function in fund2.m
global a b c d
```

```
% setting the graphical window
```

```
ind=1,id=1;

% loop for plotting some collection of trajectories
while ind<0,
n=input(' how many curves ');
t1=input(' how long curves in time ');
Plot(xmin,ymin);
axis([xmin,xmax,ymin,ymax]);
hold on;
t0=0;
for i=1:n,
[xstart,ystart]=ginput(1),
x0(1)=xstart;x0(2)=ystart;
[t,x]=ode23('fund2',t0,t1,x0);
plot(x(:,1),x(:,2));
u=size(x);
xend=x(u(1),1),
yend=x(u(1),2),
x=[ ] ;t=[ ];
x=[ ] ;t=[ ];
% to make the arrows for the directions
[qx(1),qy(1)]=ginput(1),
[qx(2),qy(2)]=ginput(1),
[qx(3),qy(3)]=ginput(1),
plot(qx,qy);

end

% do we wish more trajectories or not
ind=input(' for more curves give 1 else 0 ');
id=input(' for changing axis give 0 ');
if id==0,
xmin=input('minimal x coord in plot ');
xmax=input('maximal x coord in plot ');
ymin=input('minimal y coord in plot ');
ymax=input('maximal y coord in plot ');
axis([xmin,xmax,ymin,ymax]);
end
```

We can in fund2.m, for example, use a general linear map given below:

```
function xx1 = dysns1(t,x)
```

```

global a b c d
xx1(1) = a*x(1)+b*x(2);xx1(2) = c*x(1)+d*x(2);

The following maple file was used to examine the behaviour of a two
dimensional system. The output is seen in Appendix

# Defining the System
x0:=2.9;y0=0.15;n:=0..5;
f1:=-x^n*exp(-1/y)+1-x;
f2:= x0*2*x^n*exp(-1/y)-2*(y-y0);

# Solving for equilibria
yy:=solve(f1,y);
yy:=subs(yy,f2);

gr:=subs(yy,yy,f2);
xx1:=fsolve(gr,x=0..65..0.75);
xx2:=fsolve(gr,x=0..85..0..9);
xx3:=isolve(gr,x=0..95..0..999);
for i from 1 to 3 do
yy[i]:=evalf(subs(x=xx[i],yy));
od;

# Counting Jacobian matrix and eigenvalues and vectors at equilibria
with(linalg):
jf:=jacobian([f1,f2],[x,y]);
for i from 1 to 3 do
eigenvals(subs({x=xx[i],y=yy[i]},eval(jf)));
od;

with(DEtools): with(plots):

# Starting at a circle of small radius around the equilibria
# construct the trajectories through these points
Ant:=10;
radi:=0..0.1;
for i from 1 to 3 do
init.i:={seq([0,xx[i]+radi*sin(2*Pi/Ant*j)],
yy[i]+radi*cos(2*Pi/int*j]),j=0..int)};
od;

pp.1:=phaseportrait([f1,f2],[x,y],0..2,
init.1,x=0..2,y=-1..3,stepsize=0.05);
pp.2:=phaseportrait([f1,f2],[x,y],-4..10,
init.2,x=0..2,y=-1..3,stepsize=0.05);
pp.3:=phaseportrait([f1,f2],[x,y],0..2,
init.3,x=0..1.5,y=-1..3,stepsize=0.05);

# Starting at points on the boundary of the interesting region
# construct the trajectories starting at these points
Ant:=10;
init.4:={seq([0,0..0.01+3*j/Ant],j=0..Ant)};
init.5:={seq([0,1..1.5,0..0.01+3*j/Ant],j=0..Ant),

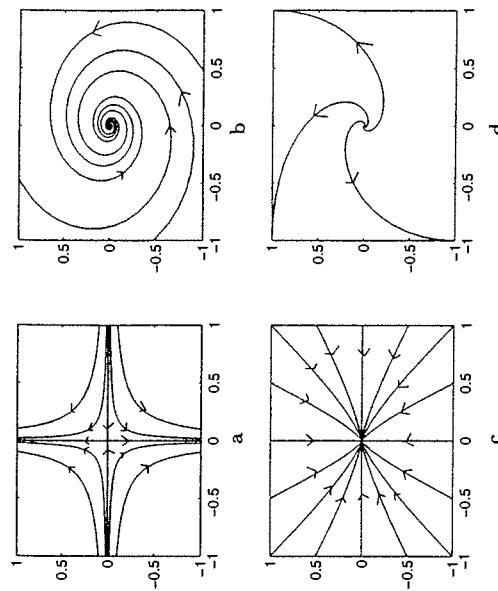
```

List 1

- 1) $x' = x, y' = 2y$
- 2) $x' = -3y, y' = 3x$
- 3) $x' = 0, y' = y$
- 4) $x' = -x, y' = y$
- 5) $x' = x - 2y, y' = 2x - y$
- 6) $x' = 3x, y' = 4y$
- 7) $x' = -x - 5y, y' = 5x - y$
- 8) $x' = 5x, y' = 4y$
- 9) $x' = 3x + 5y, y' = -5x + 3y$
- 10) $x' = -3x, y' = -2y$
- 11) $x' = 0, y' = -2y$
- 12) $x' = y, y' = -x$
- 13) $x' = -x + 3y, y' = 3x - y$
- 14) $x' = 2x, y' = 2y$
- 15) $x' = x - y, y' = x + y$
- 16) $x' = -x, y' = 0$
- 17) $x' = -x + y, y' = -x - y$
- 18) $x' = -3x, y' = -3y$
- 19) $x' = -y, y' = x$
- 20) $x' = -4x, y' = -5y$
- 21) $x' = 7x, y' = 0$

List 2

- 1) $x' = -2x - z, y' = -3y, z' = x - 2z$
- 2) $x' = 2x, y' = 3y, z' = -6z$
- 3) $x' = -2x, y' = -2y + z, z' = -y - 2z$
- 4) $x' = x, y' = -y - z, z' = y - z$
- 5) $x' = -4x, y' = 2y, z' = 4z$
- 6) $x' = 2x - 2y, y' = 2x + 2y, z' = 5z$
- 7) $x' = 7x, y' = -3y, z' = -5z$
- 8) $x' = x, y' = -4y, z' = 2z$
- 9) $x' = -x - 2z, y' = y, z' = 2x - z$
- 10) $x' = -x, y' = -2y, z' = -3z$
- 11) $x' = x - y, y' = x + y, z' = -4z$
- 12) $x' = -3x, y' = 3y, z' = -2z$



List 3

- 1) $x' = 2x - 3z, y' = 2y, z' = 3x + 2z$
- 2) $x' = -x, y' = -2y, z' = 3z$
- 3) $x' = 3x, y' = y, z' = 4z$
- 4) $x' = -4x - y, y' = x - 4y, z' = 7z$
- 5) $x' = -4x, y' = -2y, z' = -z$
- 6) $x' = -x, y' = 3y, z' = -2z$
- 7) $x' = -x, y' = 2y + x, z' = -y + 2z$
- 8) $x' = 2x, y' = y, z' = -z$
- 9) $x' = -x, y' = 6y, z' = 3z$
- 10) $x' = -x - 5y, y' = 5x - y, z' = -2z$
- 11) $x' = 3x - 2z, y' = -2y, z' = 2x + 3z$
- 12) $x' = 5x, y' = -y - 3z, z' = 3y - z$

Figure 3:

- 4) $x' = 2x + 5y, y' = y$
- 5) $x' = -x, y' = 4x + y$
- 6) $x' = 2x, y' = 4x - y$
- 7) $x' = x, y' = 3y + 5x$
- 8) $x' = 3x, y' = 4x + y$
- 9) $x' = 5y - x, y' = -2y$
- 10) $x' = 2y, y' = 8x$
- 11) $x' = 2y, y' = -4x$
- 12) $x' = 2y, y' = -x$

- ### List 4
- 1) $x' = -4x, y' = -12x - 6y$
 - 2) $x' = -2x - 2y, y' = 2y$
 - 3) $2x' = x + 10y, y' = 2y$

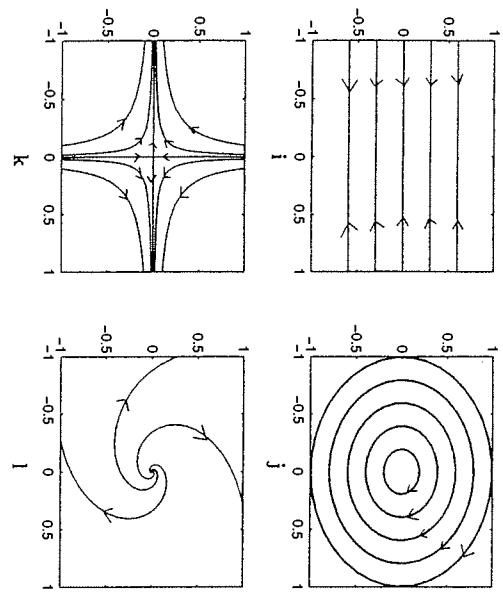


Figure 5:

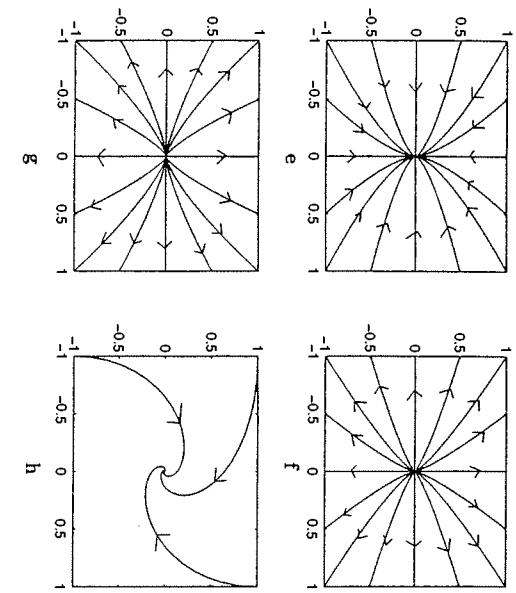


Figure 4:

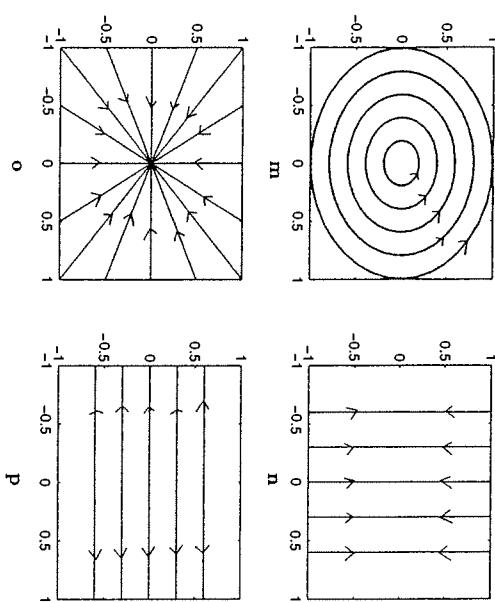
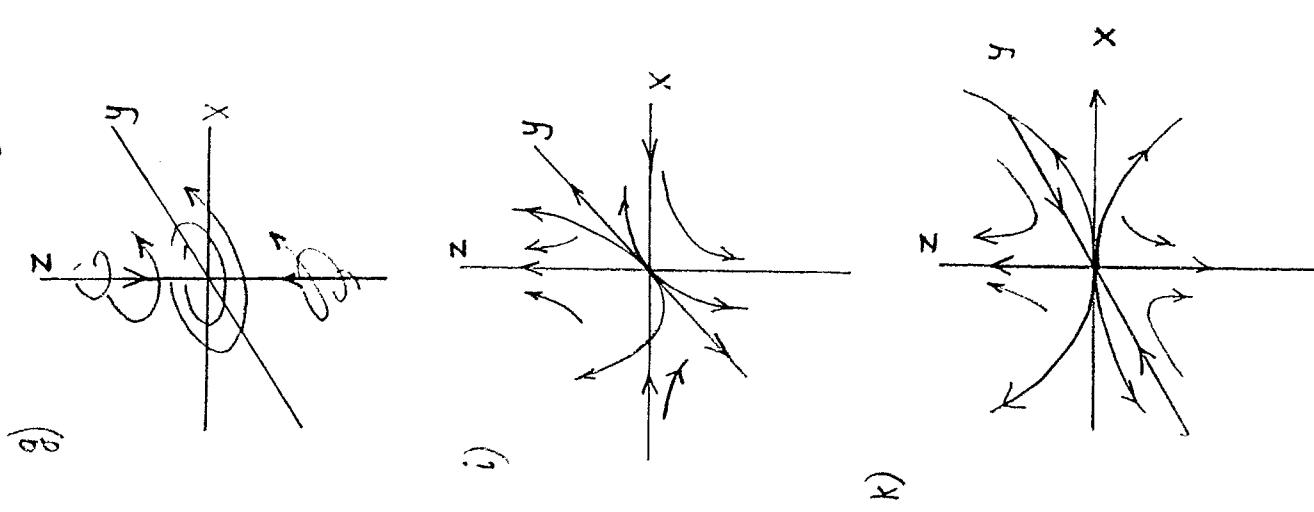
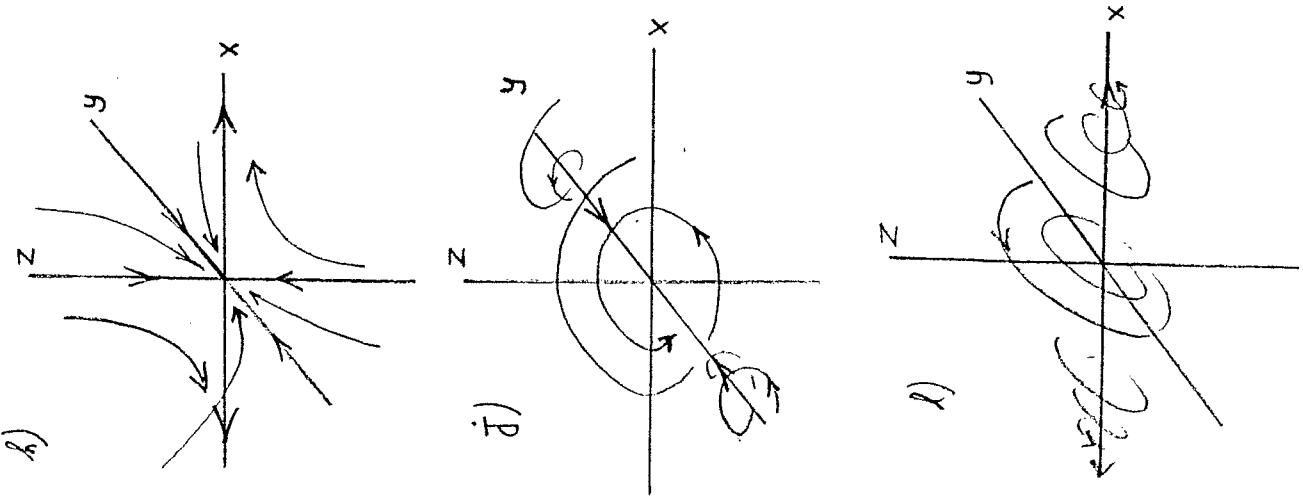
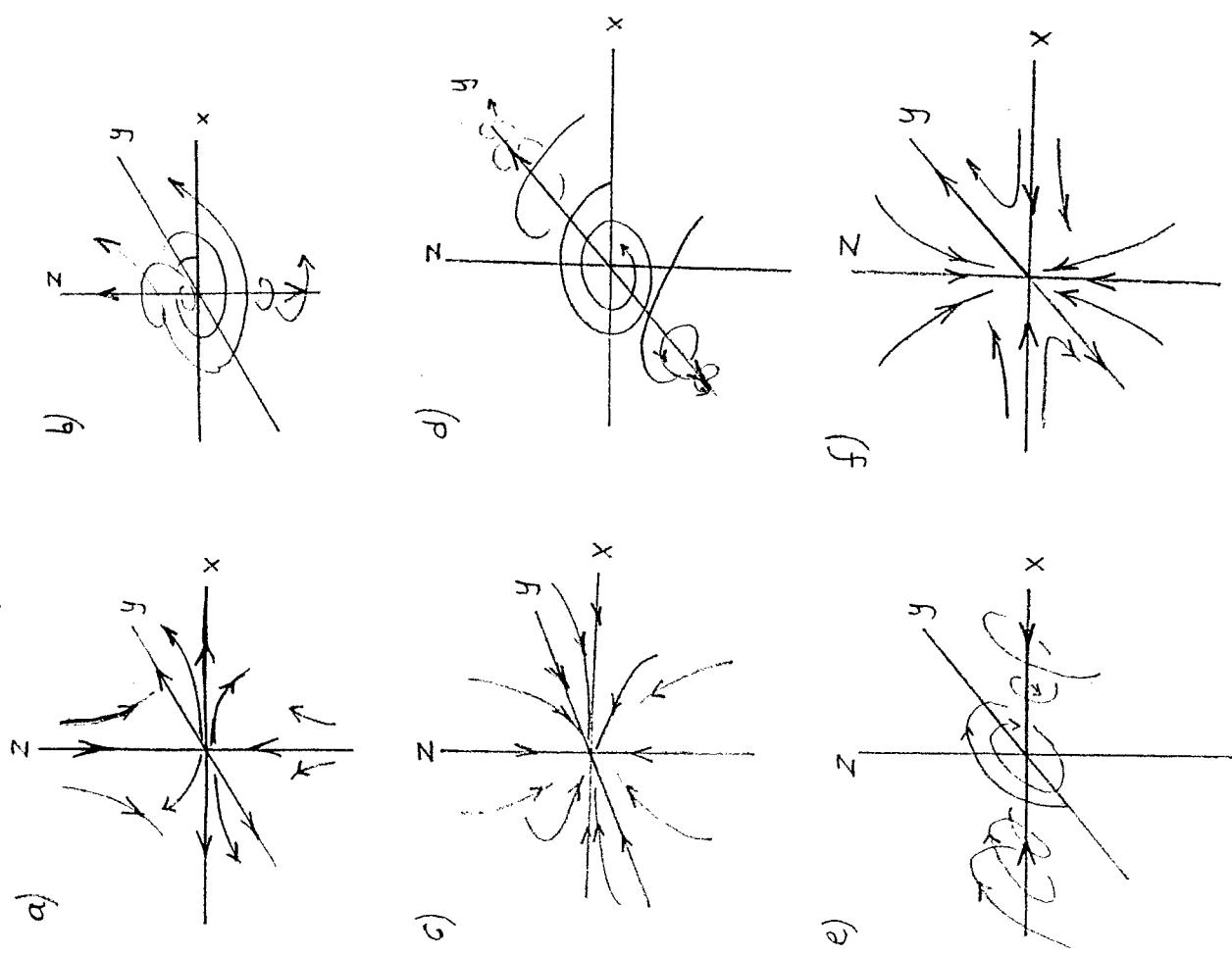


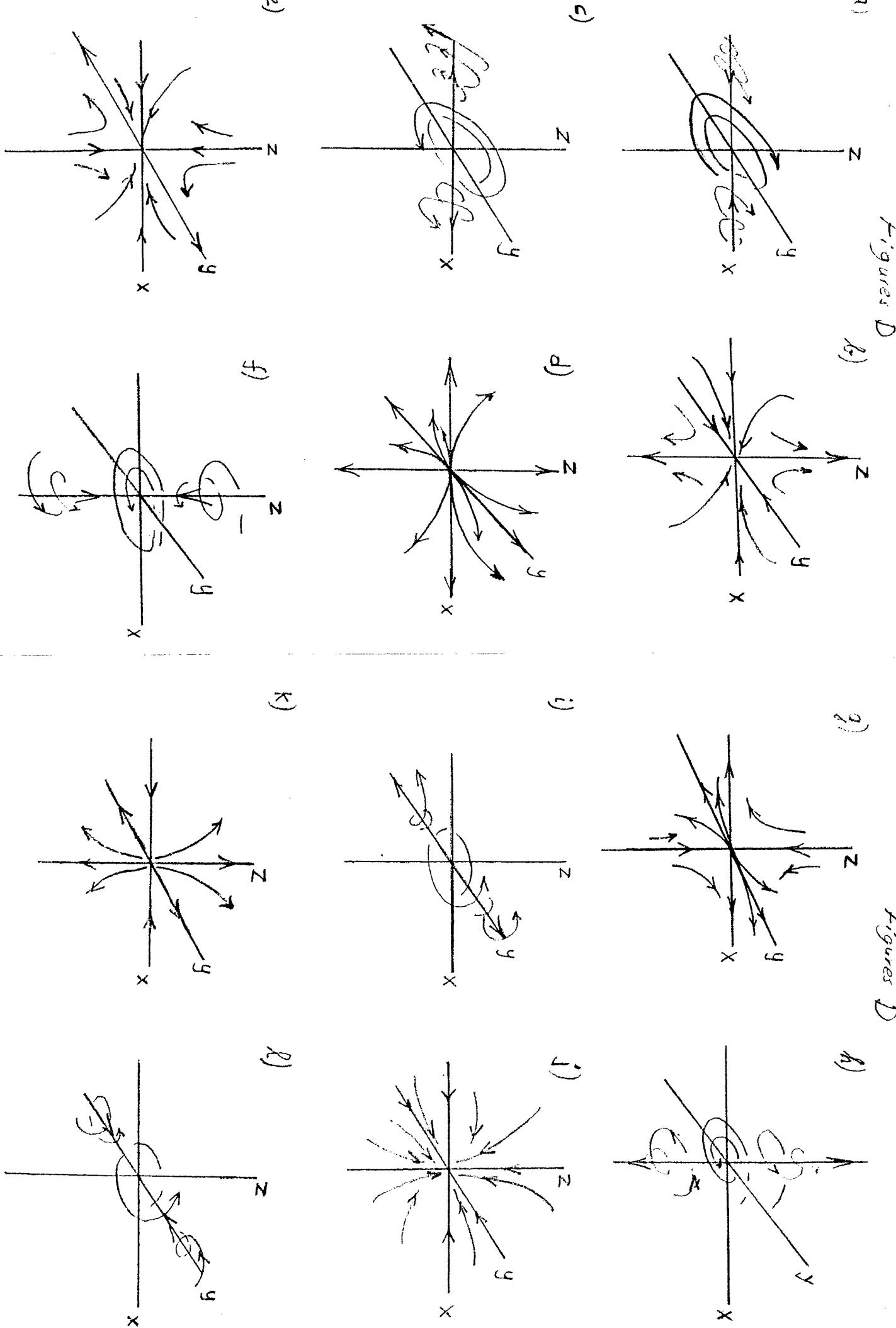
Figure 6:

Figures C



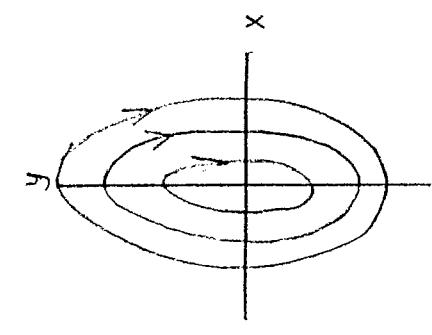
Figures C



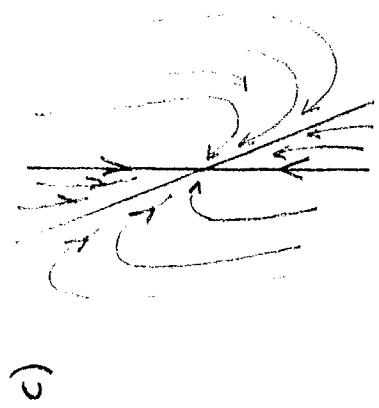


Figures E

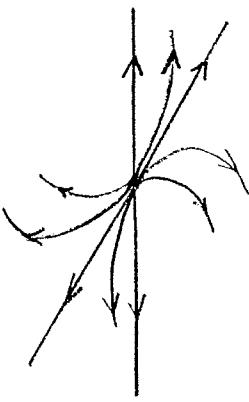
Figures E



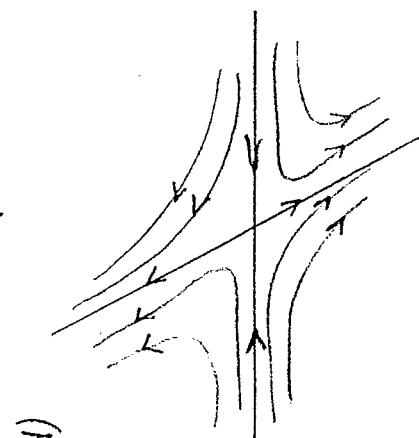
e)



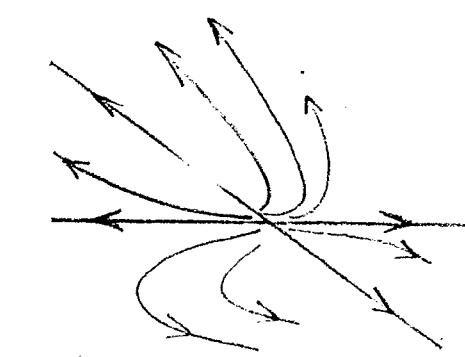
c)



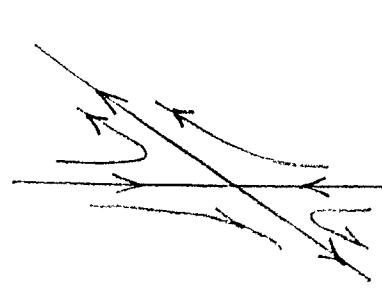
a)



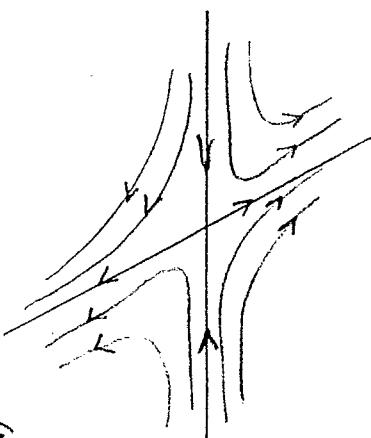
d)



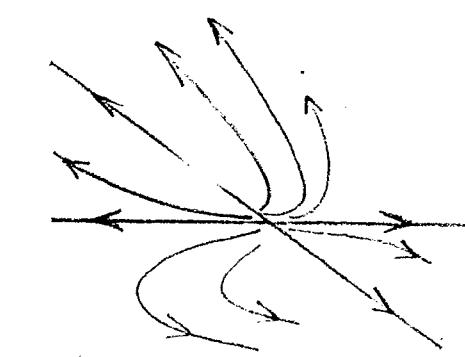
f)



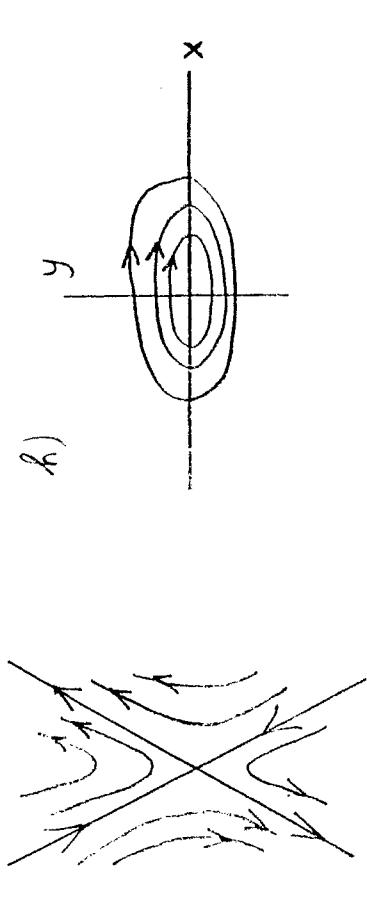
b)



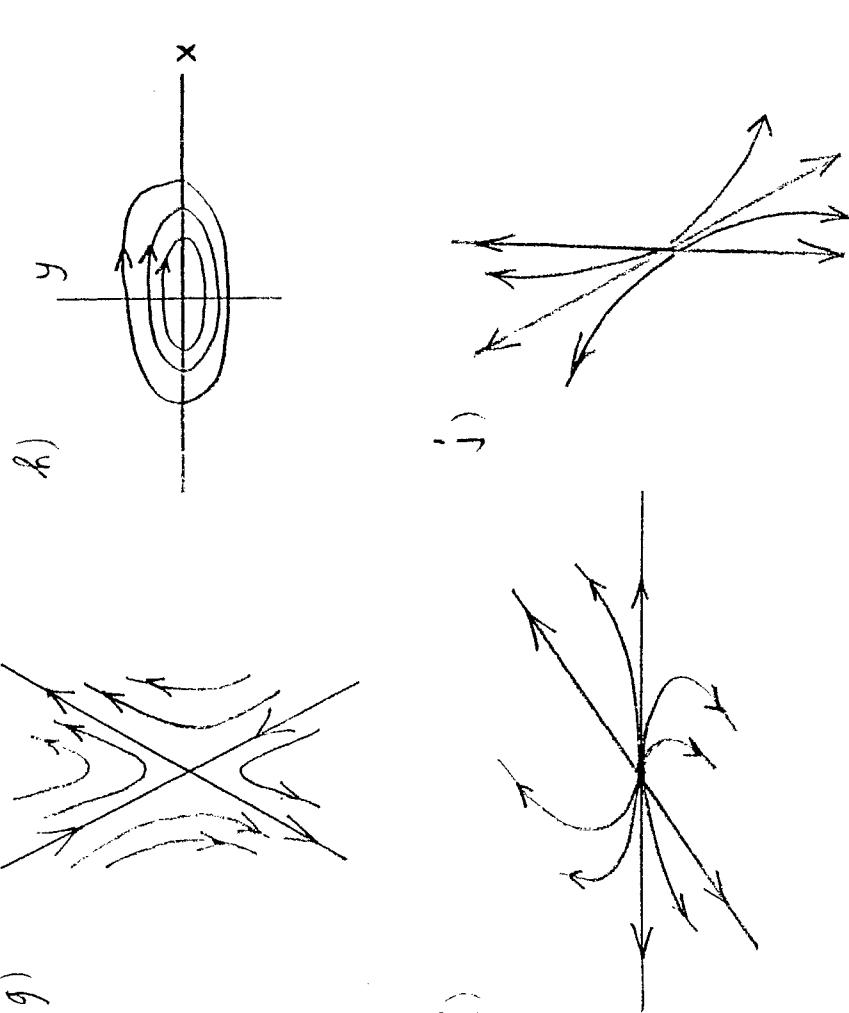
i)



j)

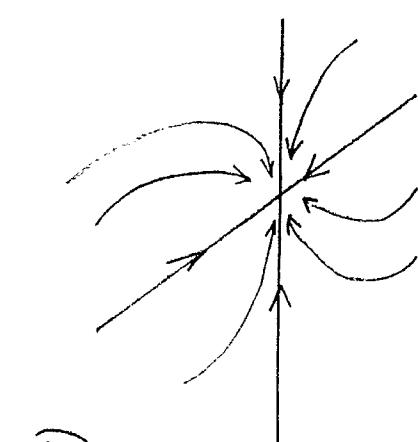


g)

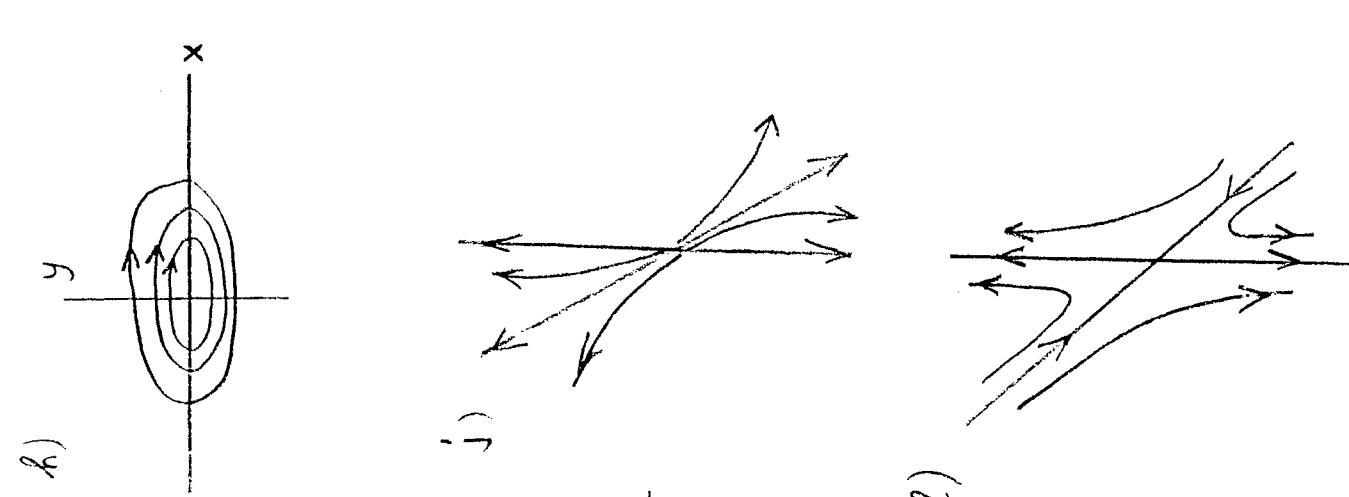


j)

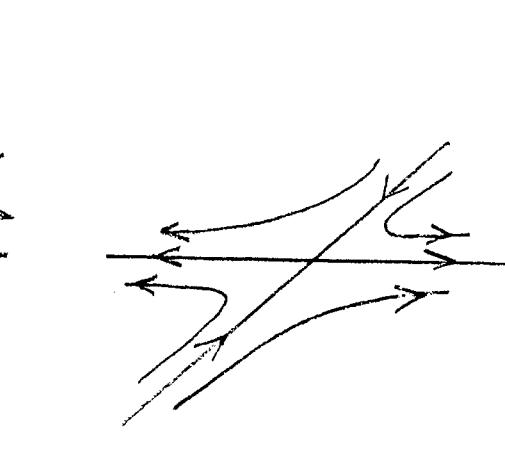
l)



k)



h)



Figures E

Appendix

Figure F1 Figure F2

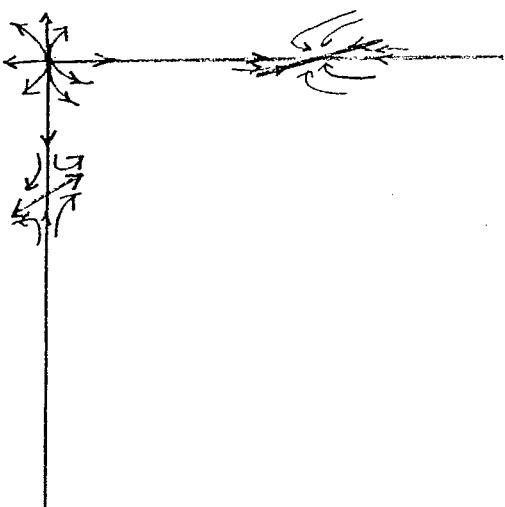
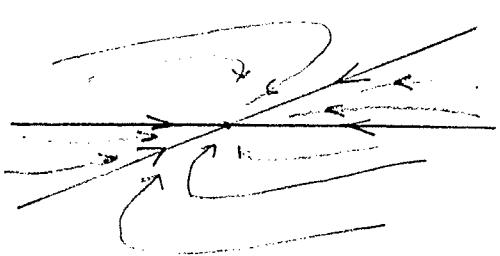
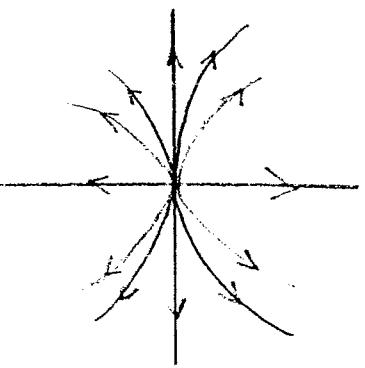


Figure F3

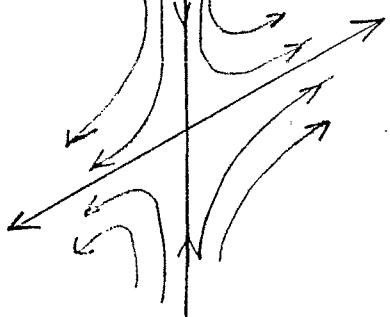


Figure F4

```
> read('u22b1.ms');
```

We set some parameters specifying the system:

$$x0 := 2.9$$

$$y0 := .15$$

$$n := .5$$

Here is the two dimensional system, f1 represents x' and f2 y' :

$$f1 := -x^5 e^{-y} + 1 - x$$

$$f2 := 5.8 x^5 e^{-y} - 2 y + .30$$

If we solve $x' = 0$ with resp to y we get:

$$yy := -1. \frac{1}{\ln \left(-1. \frac{-1.+x}{\sqrt{x}} \right)}$$

If this solution is substituted into the expression for y' we get:

$$gr := 5.8 x^5 e^{1. \ln \left(-1. \frac{-1.+x}{\sqrt{x}} \right)} + 2. \frac{1}{\ln \left(-1. \frac{-1.+x}{\sqrt{x}} \right)} + .30$$

The roots of gr are the x-coordinates of the equilibria:

$$xx1 := -6664158693$$

$$xx2 := .8945194734$$

$$xx3 := .9984575641$$

Substituting into the yy-expression we get the corresponding y-coordinates of the equilibria:

$$yy1 := 1.117393978$$

$$yy2 := 4558935271$$

$$yy3 := 1.544730639$$

The Jacobian matrix for the system is:

$$Jf := \begin{bmatrix} -\frac{1}{x^5} & -\frac{x^5 e^{-y}}{y^2} \\ -5 \frac{e^{-y}}{x^5} - 1 & \frac{1}{x^5} \\ 2.90 \frac{e^{-y}}{x^5} & 5.8 \frac{x^5 e^{-y}}{y^2} - 2 \end{bmatrix}$$

The eigenvalues and eigenvectors of the first equilibria are given by:
 (first number is the eigenvalue, second its multiplicity=1, third the corresponding eigenvector)

$$[-.8503391380 + .4773718463 \ I, 1, \{ [-.3288506865 - .2755117817 \ I] \}],$$

[-.8503391380 - .4773718463 \ I, 1, \{ [-.3288506865 + .2755117817 \ I] \}]

As You see it is a stable focus.

The eigenvalues and eigenvectors of the second equilibria are given by:

$$[-.968177954, 1, \{ [-.9843757037 .1760808731] \}],$$

$$[.8527822766, 1, \{ [.2831288277 -.1066517151] \}]$$

As You see it was a saddle.

The eigenvalues and eigenvectors of the third equilibria are given by:

$$[-1.001236599, 1, \{ [9.974314781 .07162713519] \}],$$

$$[-1.624623589, 1, \{ [.1039586894 1.003321680] \}]$$

It was a stable node.

Now we choose some initial conditions around the equilibria and find the forwards and backwards (there are ten of them on distance 0.01 from the equilibrium):

$$radie := .01$$

$$Ant := 10$$

$$\begin{aligned} init1 &:= \{ [0, .6664158693 + .002500000000 \%1, 1.119893978 + .002500000000 \sqrt{5}], \\ &\quad [0, .6664158693 + .002500000000 \%2, 1.114893978 + .002500000000 \sqrt{5}], \\ &\quad [0, .6664158693, 1.107393978], \\ &\quad [0, .6664158693 - .002500000000 \%2, 1.114893978 + .002500000000 \sqrt{5}], \\ &\quad [0, .6664158693 - .002500000000 \%1, 1.119893978 + .002500000000 \sqrt{5}], \\ &\quad [0, .6664158693 - .002500000000 \%1, 1.114893978 - .002500000000 \sqrt{5}], \\ &\quad [0, .6664158693 - .002500000000 \%2, 1.119893978 - .002500000000 \sqrt{5}], \\ &\quad [0, .6664158693 + .002500000000 \%2, 1.1119893978 - .002500000000 \sqrt{5}], \\ &\quad [0, .6664158693 + .002500000000 \%1, 1.1114893978 - .002500000000 \sqrt{5}], \\ &\quad [0, .6664158693, 1.127393978] \} \end{aligned}$$

$$\%1 := \sqrt{5 - \sqrt{5}}$$

$$\%2 := \sqrt{5 + \sqrt{5}}$$

We also choose some initial conditions on the boundary of the square where x is between 0 and 1.5 and y is between 0 and 3 and count the trajectory only in forward direction:

$$Ant := 10$$

$$\begin{aligned} init4 &:= \{ [0, 0, .9100000000], [0, 0, .6100000000], [0, 0, .3100000000], [0, 0, .01], \\ &\quad [0, 0, 3.01], [0, 0, 2.7100000000], [0, 0, 2.4100000000], [0, 0, 2.1100000000], \} \end{aligned}$$

```

[0, 0, 1.81000000], [0, 0, 1.51000000], [0, 0, 1.21000000]
[0, 1.5, .376666667], [0, 1.5, .370000000], [0, 1.5, .363333333],

```

```

[0, 1.5, .356666667], [0, 1.5, .301], [0, 1.5, .271000000], [0, 1.5, .241000000],
[0, 1.5, .211000000], [0, 1.5, 1.81000000], [0, 1.5, 1.51000000],
[0, 1.5, 1.21000000], [0, 1.5, .910000000], [0, 1.5, .610000000],
[0, 1.5, .310000000]

```

```

init5 := {[0, 1.5, .35], [0, 1.5, .01], [0, 1.5, .390000000], [0, 0, 1.21000000]

```

```

[0, 1.5, .300000000, .01], [0, 1.350000000, .01], [0, 1.200000000, .01],
[0, 1.050000000, .01], [0, 900000000, .01], [0, 750000000, .01],
[0, 600000000, .01]}

```

```

init6 := {[0, 0, 3], [0, .750000000, 3], [0, .600000000, 3], [0, .450000000, 3],

```

```

[0, .300000000, 3], [0, 1.5, 3], [0, 1.350000000, 3], [0, 1.200000000, 3],
[0, 1.050000000, 3], [0, 900000000, 3], [0, 1.500000000, 3]}

```

```

init7 := {[0, 0, .8], [0, 1.5, .2], [0, 0, 1.000000000], [0, 0, .966666667],

```

```

[0, 0, .933333334], [0, 0, .900000000], [0, 0, .866666667], [0, 0, .833333333],
[0, 1.5, .400000000], [0, 1.5, .350000000], [0, 1.5, .300000000],

```

```

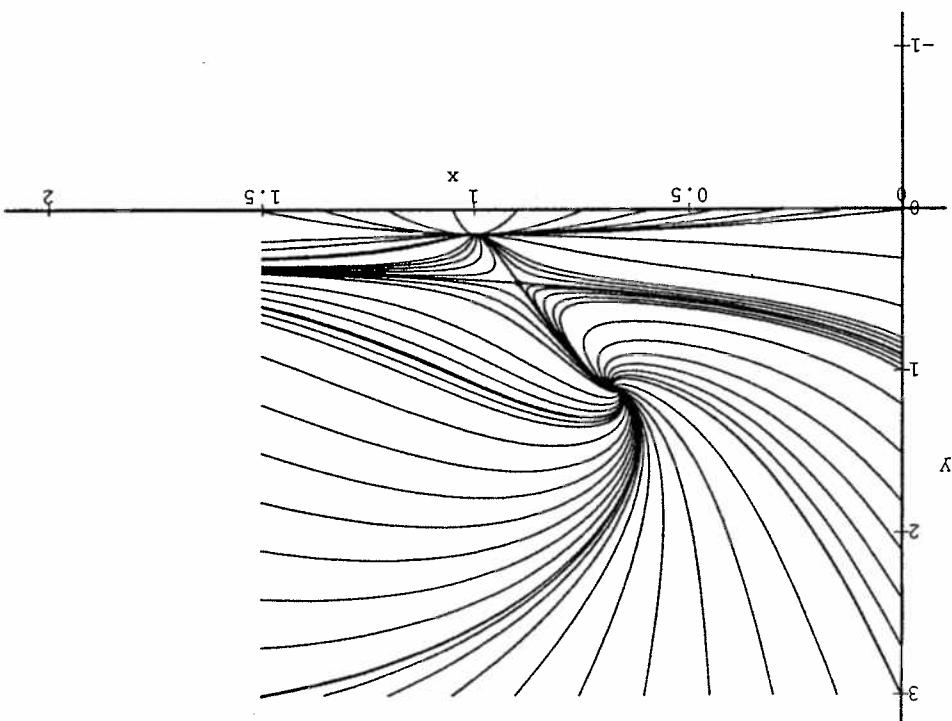
[0, 1.5, .250000000], [0, 1.5, .600000000], [0, 1.5, .550000000],
[0, 1.5, .500000000], [0, 1.5, .450000000], [0, 1.5, .400000000]

```

Now we can plot the phase portrait in the interesting region which is the square mentioned above.

The plot is seen in the figure. Convince Yourself that the results coincide with the analytical results above.

To plot the zero-isoclines You can use the Maple command `implicitplot`.



Examples of phase portraits

Electrical circuit with tunnel diode

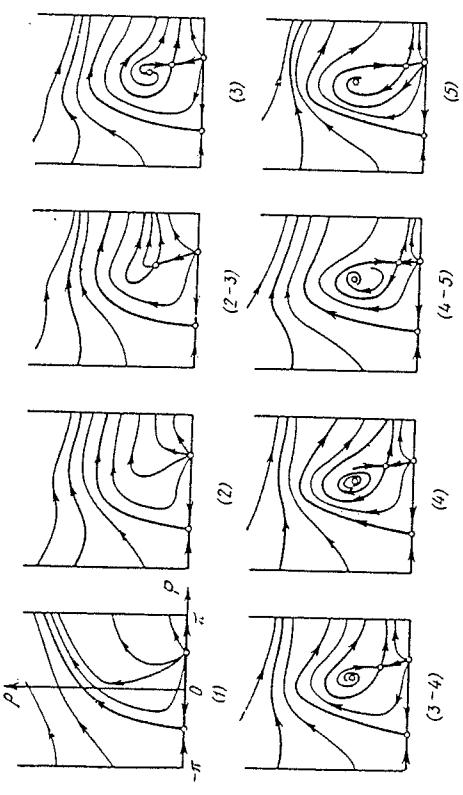
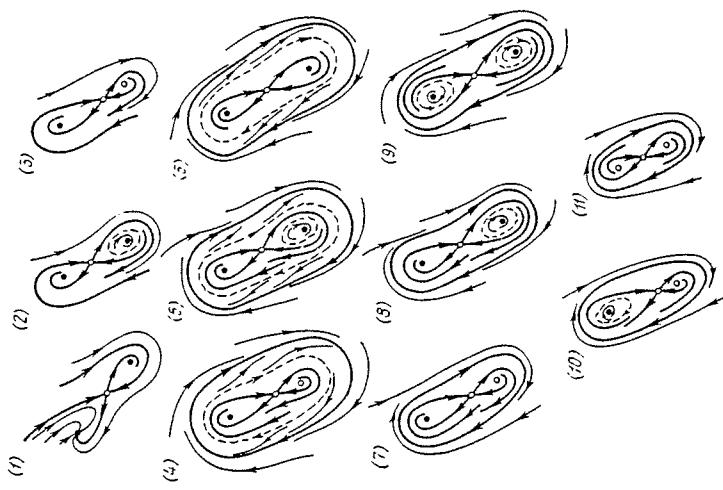
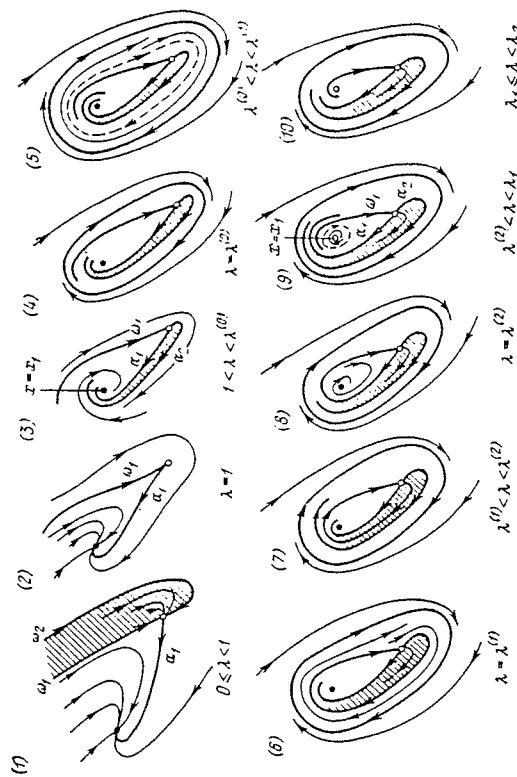
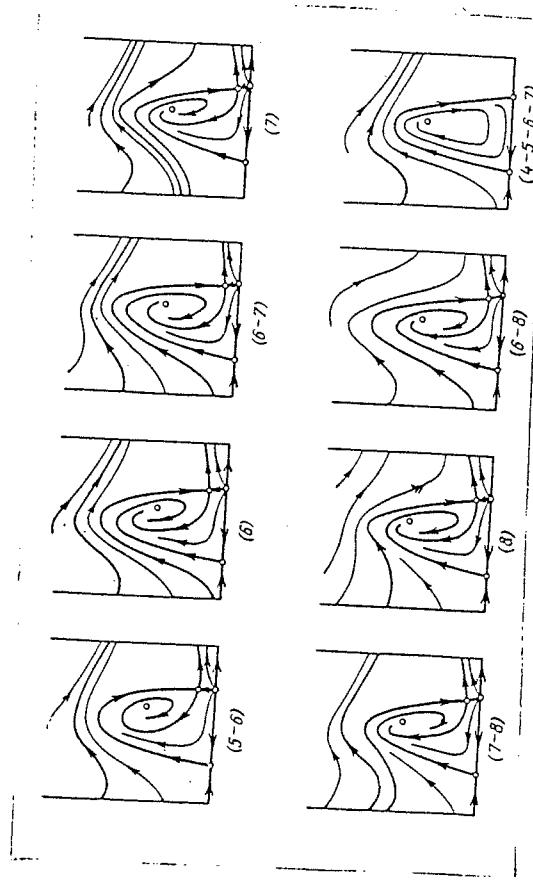
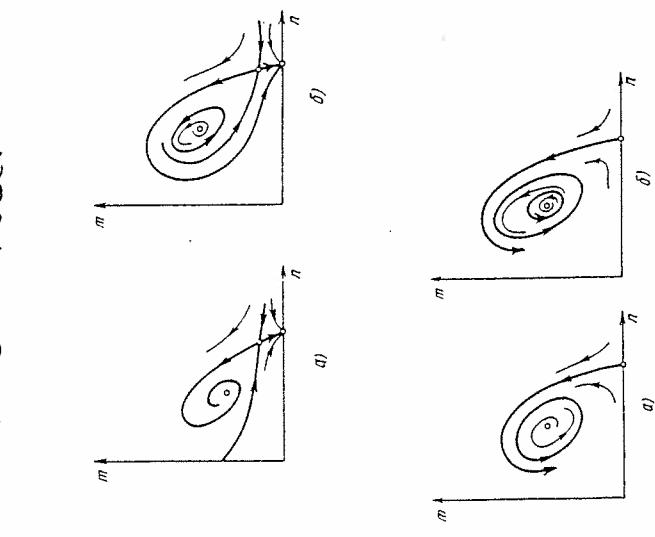
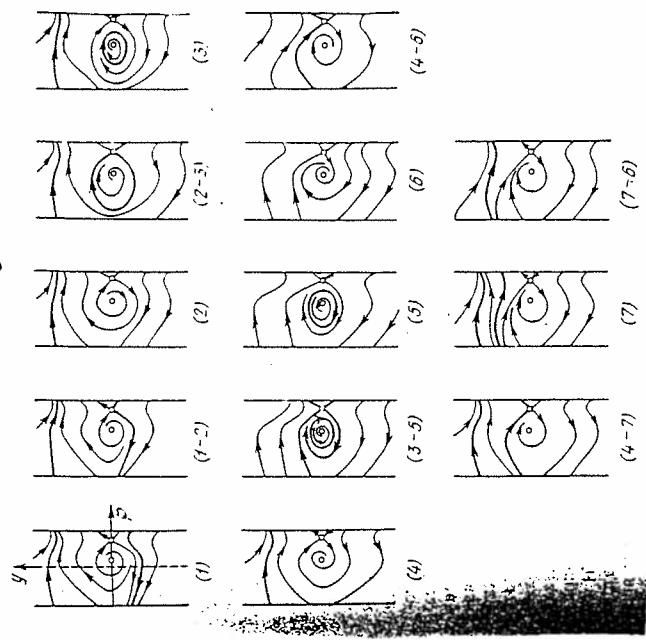


fig. Баютин Н.К. . . методы и приемы изучения ...

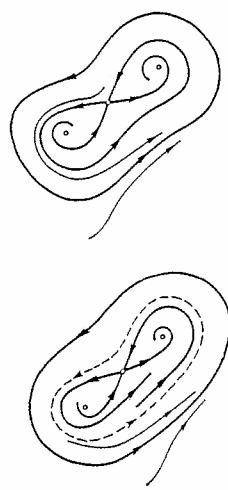
Laser model



Synchronous generators



Chemical reactor



Frequency tuning

