

# 1 Constructing Phase Portraits

## What can be done?

As well known most of non-linear ordinary differential equations cannot be solved explicitly. So any known to You expression of elementary functions, for example, combinations of polynoms, trigonometric, logarithmic functions etc cannot be the solution of the equations. We cannot find any finite expression of elementary functions as the solution of the equations. All the solutions You have learnt in the ground course at a technical university of linear equations, separable equations etc are not more than good approximations. Simulations are, of course, also approximations although usually good. But to plan the simulations and especially in high dimensional systems or systems with many parameters it is more effective or often practically necessary to know something from Mathematics about the behaviour of the solutions. This is to try to get a general geometric picture about the solutions. And that is what is done when You construct the phase portrait. If the phase portrait is plotted the qualitative behaviour of the solutions can be seen from it at once. From the plot we immediately can see where the trajectory starting at a given point is approximately going to be. The phase portrait is especially easy to plot in the two dimensional case.

## What are we going to do?

Our aim now is to demonstrate some of the very central methods in constructing the phase portrait for a two dimensional system by applying them to some easy examples. The examples are chosen so that all calculations can be done with minimal knowledge of analysis and linear algebra and are technically as short as we manage to do.

As good examples we have chosen the the following two two-dimensional systems

$$(1.1) \quad x' = x(2 - x - y) \quad y' = y(6 - 2x - y)$$

and

$$(1.2) \quad x' = x(2 - x - y) \quad y' = y(3 - 2x - y).$$

We start with these systems because some very central ideas in constructing the phase portrait are easy to see in these cases. We are going to construct the phase portrait only in the region  $x, y \geq 0$ . At the end we give some examples of phase portraits from technical applications.

Before we start with constructing the phase portrait itself we recall some important facts playing a central role in the constructing process.

## What is phase portrait?

We start by repeating the definition of the phase portrait. Let  $x' = f(x)$   $x \in B^n$  be a system of differential equations. (It is called autonomous when the right side does not depend on time.) It is known (Theorem of Existence and Uniqueness) that if  $f$  is continuous and differentiable in a neighbourhood of a point  $x_0$  there exists a unique solution  $\varphi$  with initial conditions  $\varphi(0) = x_0$  in a neighbourhood of  $x_0$  defined for a time interval containing 0. The existence and uniqueness of solutions imply that the subspace of  $B^n$  where the system is defined can be divided into nonintersecting curves. By the *phase portrait* we mean a collection of solution curves with time direction demonstrating the geometrical behaviour of the solutions.

### Linearizations

We also repeat how to construct the phase portrait locally around some equilibrium. Suppose  $x_0$  is an equilibrium for a system  $x' = f(x)$  and suppose that the real parts of the eigenvalues of the Jacobian of  $f$  at the point  $x_0$  are non-zero. Then according to a Theorem of Grobman-Hartman there is a change of coordinates (this transform is not necessary differentiable) in a neighbourhood of the equilibrium such that in the new coordinates the system takes the linearized form  $x' = Ax$ , where the matrix  $A$  is the Jacobian of  $f$  at  $x_0$ . This theorem is very useful and of great help for constructing the phase portrait. The phase portrait of a linear system is easy to construct and depends on the signs of the eigenvalues or the real parts of them. So if we know the eigenvalues of the Jacobian at the equilibrium we can often construct the phase portrait in a neighbourhood of the equilibrium.

(\* An equilibrium is a sink if the real parts of all eigenvalues of the Jacobian at the equilibrium point are negative. If all the real parts are positive it is a source and if some parts are negative and other positive it is called a saddle. Any solution in some neighbourhood of a sink tends to it for time  $t \rightarrow \infty$ . (It can also be called a simple attractor). Any solution in some neighbourhood of a source except for the source itself leaves the neighbourhood and tends to it for  $t \rightarrow -\infty$ . Let a saddle have  $k$  eigenvalues with negative real part and  $n-k$  with positive real part. In some neighbourhood of the saddle there exists a  $k$ -dimensional subspace where the solutions tend to the saddle for  $t \rightarrow \infty$  and a  $n-k$ -dimensional subspace where the solutions tend to the saddle for  $t \rightarrow -\infty$ . All other solutions in the neighbourhood of the saddle leave it both for increasing and decreasing time. \*)

To construct the phase portrait in the two dimensional case around the equilibria is especially easy. In this case we repeat the phase portrait for the linear systems for non-zero real parts of the eigenvalues. Some of the central phase portraits are seen in figure 1. We have a source if both eigenvalues are real and positive (in this case the equilibrium is called an unstable node) or if there is a pair of complex eigenvalues with positive real parts (now the equilibrium is called an unstable focus). Analogously we have a sink if both eigenvalues are real and negative (called stable node) or there is a pair of complex eigenvalues with negative real parts (called stable focus). If one eigenvalue is greater than zero and the other is less than zero we have a saddle (this happens exactly when the determinant of the Jacobian is less than zero).

## Planning the construction of the phase portrait

Now we can start to construct the phase portrait for our systems. The first thing we do is to calculate the equilibria. Here You need to know how to solve non-linear systems of equations. These can, of course, not always be solved explicitly and must then be solved numerically. In our case however the solutions are simple and nice. After You have got the equilibria You have to find out the type of the equilibria to construct the phase portrait locally around each of the equilibria. As You know the type of the equilibria is determined by the sign of the real part of the eigenvalues of the Jacobian at the equilibrium. Here You need to repeat Your knowledge from linear algebra. When the phase portrait is constructed locally around the equilibria we want to construct it globally so we can see what happens far from equilibria. To do that it is of great importance to know the sign of  $x'$  and  $y'$  in different regions. When, for example,  $x' > 0$  the  $x$ -coordinate is increasing, when  $x' < 0$  it is decreasing and analogously for the  $y$ -coordinate. It is especially important to know where the sign of  $x'$  and  $y'$  changes, that is to plot the curves  $x' = 0$  and  $y' = 0$ . When we have done all these things we will see that in this case that we can conclude the whole behaviour of the solutions in  $x, y \geq 0$ . This is, of course, not always the case, especially when there are periodic solutions present. The last situation is more complicated and we do not touch it here. (Anyhow it is an important result that there can be no chaotic solutions in the two dimensional case).

### Solving for equilibria

Let us start to solve for the equilibria. We consider first the system (1.1). The equilibria are solutions of the systems of equations  $x' = 0$ ,  $y' = 0$ . These equations imply that one of  $x = 0$  or  $x + y = 2$  must be satisfied and one of  $y = 0$  or  $2x + y = 6$  must be satisfied. Thus  $x = y = 0$  is an equilibrium.  $x = 0$  and  $2x + y = 6$  gives the equilibrium  $x = 0, y = 6$ .  $y = 0$  and  $x + y = 2$  gives the equilibrium  $x = 2, y = 0$ . The system of equations  $x + y = 2$ ,  $2x + y = 6$  does not have non-negative solutions. Thus we have got three equilibria and start to determine the type of them.

### Local phase portraits

The Jacobian for the system is

$$J = \begin{pmatrix} 2 - 2x - y & -x \\ -2y & 6 - 2x - 2y \end{pmatrix}.$$

At the origin it is

$$J = \begin{pmatrix} 2 & 0 \\ 0 & 6 \end{pmatrix}$$

and clearly the eigenvalues are both positive and we have an unstable node. The eigenvectors are  $(0,1)$  and  $(1,0)$ . The local phase portrait around the origin is shown in figure 2a. The Jacobian at the the second equilibrium  $x = 0, y = 6$  is

$$J = \begin{pmatrix} -4 & 0 \\ -12 & -6 \end{pmatrix}.$$

It has one eigenvalue -4 with eigenvector  $(1,6)$  and another eigenvalue -6 with eigenvector  $(0,1)$ . Because both eigenvalues are less than zero it is a

stable node. The phase portrait around the equilibrium is shown in figure 2b. The Jacobian at the the third equilibrium  $x = 2, y = 0$  is

$$J = \begin{pmatrix} -2 & -2 \\ 0 & 2 \end{pmatrix}.$$

It has one eigenvalue -2 with eigenvector  $(1,0)$  and another eigenvalue 2 with eigenvector  $(-1,2)$ . Because one eigenvalue is less than zero and the other greater it is a saddle. The phase portrait around the equilibrium is shown in figure 2c.

Now we can plot a figure where the phase portrait is given around all the three equilibria in the region  $x, y \geq 0$ . This is seen in figure 2d.

#### Global phase portrait

At the next stage we calculate the regions for the positivness or negativeness of  $x'$ . On the common boundary of these regions  $x' = 0$ . This boundary is given by the lines  $x = 0$  and  $x + y = 2$  intersected with  $x, y \geq 0$ . In the region  $x + y < 2, x > 0$   $x'$  is positive. In the region  $x + y > 2, x > 0$  it is negative. (See figure 3)

We continue by calculating the sign of  $y'$  in different regions.  $y' = 0$  gives  $y = 0$  or  $2x + y = 6$ . In the region  $2x + y < 6, y > 0$   $y'$  is positive and in the region  $2x + y > 6, y > 0$  it is negative. (See figure 4).

Now we try to connect the results above on the signs of  $x'$  and  $y'$ . We get the result: seen in figure 5. In the region below the line  $x + y = 2$  both signs are positive and both  $x$  and  $y$  are increasing. Thus the trajectories move in direction up and right as is seen by the arrows in the figure. Above  $x + y = 2$  but below the line  $2x + y = 6$   $x'$  is negative and  $y'$  is positive. Thus the trajectories here move left and end up according to the arrows in the figure. Above the line  $2x + y = 6$  both  $x'$  and  $y'$  are less than zero and both  $x$  and  $y$  decrease. On the line  $x + y = 2 (x, y > 0)$  we have  $x' = 0$  but  $y' > 0$  and thus the trajectories go vertically up here. Because both  $x$  and  $y$  are increasing below the line any trajectory starting in  $x, y > 0$  below  $x + y = 2$  will reach this line, intersect it and then remain above it.

On the line  $2x + y = 6 (x, y > 0)$  we have  $y' = 0$  and  $x' < 0$ . Thus all the trajectories go horizontally to the left here. Because both  $x$  and  $y$  are decreasing above the line  $2x + y = 6$  any trajectory starting above that line will hit it, intersect it and remain below it. We conclude that any trajectory starting in  $x, y > 0$  will enter the region between the lines  $x + y = 2$  and  $2x + y = 6$  and remain there. In the latter region any trajectory moves to the left and up and will thus tend to the equilibrium  $x = 0, y = 6$  which is thus attracting everything in the region  $x, y > 0$ . On the  $x$ -axis  $y' = 0$  and  $x$  is increasing for  $x < 2$  and decreasing for  $x > 2$ . Thus any trajectory starting on the  $x$ -axis above  $x = 2$  will tend to  $x = 2$  from above and any trajectory starting on the  $x$ -axis between 0 and 2 will tend to  $x = 2$  from below. On the  $y$ -axis  $x' = 0$  and  $y$  is increasing for  $y < 6$  and decreasing for  $y > 6$ . Thus any trajectory starting on the  $y$ -axis above  $y = 6$  will tend to 6 from above and any trajectory starting on the  $y$ -axis between 0 and 6 will tend to  $y = 6$  from below. The whole phase portrait is illustrated in figure 6.

The second example

#### Solving for equilibria

We now consider the system (1.2). Again the equilibria are solutions of the systems of equations  $x' = 0, y' = 0$ . These equations imply that one of  $x = 0$  or  $x + y = 2$  must be satisfied and one of  $y = 0$  or  $2x + y = 3$  must be satisfied. Thus  $x = y = 0$  is an equilibrium.  $x = 0$  and  $2x + y = 3$  gives the equilibrium  $x = 0, y = 3$ .  $y = 0$  and  $x + y = 2$  gives the equilibrium  $x = 2, y = 0$ . The system of equations  $x + y = 2, 2x + y = 3$  has the solution  $x = y = 1$ . Thus we have now got four equilibria and start to determine the type of them.

#### Local phase portrait

The Jacobian for the system is

$$J = \begin{pmatrix} 2 - 2x - y & -x \\ -2y & 3 - 2x - 2y \end{pmatrix}.$$

At the origin it is

$$J = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

and clearly the eigenvalues are both positive and we have an unstable node. The eigenvectors are  $(0,1)$  and  $(1,0)$ . The local phase portrait around the origin is shown in figure 7a. The Jacobian at the the second equilibrium  $x = 0, y = 3$  is

$$J = \begin{pmatrix} -1 & 0 \\ -6 & -3 \end{pmatrix}.$$

It has one eigenvalue -1 with eigenvector  $(1,3)$  and another eigenvalue -3 with eigenvector  $(0,1)$ . Because both eigenvalues are less than zero it is a stable node. The phase portrait around the equilibrium is shown in figure 7b. The Jacobian at the the third equilibrium  $x = 2, y = 0$  is

$$J = \begin{pmatrix} -2 & -2 \\ 0 & -1 \end{pmatrix}.$$

It has one eigenvalue -2 with eigenvector  $(1,0)$  and another eigenvalue -1 with eigenvector  $(-2,1)$ . Again because both eigenvalues are less than zero it is a stable node. The phase portrait around the equilibrium is shown in figure 7c. The Jacobian at the the fourth equilibrium  $x = 1, y = 1$  is

$$J = \begin{pmatrix} -1 & -1 \\ -2 & -1 \end{pmatrix}.$$

It has one eigenvalue  $-1 - \sqrt{2}$  with eigenvector  $(1, \sqrt{2})$  and another eigenvalue  $-1 + \sqrt{2}$  with eigenvector  $(-1, \sqrt{2})$ . Because one eigenvalue is less than zero and the other greater it is a saddle. The phase portrait around the equilibrium is shown in figure 7d.

Now we can plot a figure where the phase portrait is given around all the three equilibria in the region  $x, y \geq 0$ . This is seen in figure 7e.

### Global phase portrait

At the next stage we calculate the regions for the positivity or negativity of  $x'$ . On the common boundary of these regions  $x' = 0$ . This boundary is given by the lines  $x = 0$  and  $x + y = 2$  intersected with  $x, y \geq 0$ . In the region  $x + y < 2$ ,  $x > 0$   $x'$  is positive. In the region  $x + y > 2$ ,  $x > 0$  it is negative. (See figure 8).

We continue by calculating the sign of  $y'$  in different regions.  $y' = 0$  gives  $y = 0$  or  $2x + y = 3$ . In the region  $2x + y < 3$ ,  $y > 0$   $y'$  is positive and in the region  $2x + y > 3$ ,  $y > 0$  it is negative. (See figure 9). Now we try to connect the results above on the signs of  $x'$  and  $y'$ . We get the result seen in figure 10. In the region below both lines  $x + y = 2$  and  $2x + y = 3$  both signs are positive and both  $x$  and  $y$  are increasing. Thus the trajectories move in direction up and right as is seen by the arrows in the figure. Above  $x + y = 2$  but below the line  $2x + y = 6$   $x'$  is negative and  $y'$  is positive. Thus the trajectories here move to the left and up according to the arrows in the figure. Above  $2x + y = 3$  but below the line  $x + y = 2$   $y'$  is negative and  $x'$  is positive. Thus the trajectories here move to the right and down according to the arrows in the figure. Above both the lines  $2x + y = 6$  and  $x + y = 2$  both  $x'$  and  $y'$  are less than zero and both  $x$  and  $y$  decrease. On the line  $x + y = 2$  ( $x, y > 0$ ) below the line  $2x + y = 3$  we have  $x' = 0$  but  $y' > 0$  and thus the trajectories go vertically up here. On the line  $x + y = 2$  above the line  $2x + y = 3$  we have  $x' = 0$  but  $y' < 0$  and thus the trajectories go vertically down here. On the line  $2x + y = 3$  above the line  $x + y = 2$  we have  $y' = 0$  but  $x' < 0$  and thus the trajectories move horizontally to the left here. On the line  $2x + y = 3$  below the line  $x + y = 2$  we have  $y' = 0$  but  $x' > 0$  and thus the trajectories move horizontally to the right here. Because both  $x$  and  $y$  are increasing below the two lines any trajectory starting in  $x, y > 0$  below  $x + y = 2$  and  $2x + y = 3$  will reach one of these lines. If it intersects the line  $x + y = 2$  it will thereafter remain below the line  $2x + y = 3$  and tend to the equilibrium  $x = 0, y = 3$ . If it intersects the line  $2x + y = 3$  it will thereafter remain below the line  $x + y = 2$  and tend to the sink  $x = 0, y = 6$ .

We conclude that any trajectory starting in  $x, y > 0$  will enter one of the regions between the lines  $x + y = 2$  and  $2x + y = 3$  and remain there. In the upper region any trajectory moves to the left and up and will thus tend to the equilibrium  $x = 0, y = 3$ . In the lower region any trajectory moves to the right and down and will thus tend to the equilibrium  $x = 2, y = 0$ . On the  $x$ -axis  $y' = 0$  and  $x$  is increasing for  $x < 2$  and decreasing for  $x > 2$ . Thus any trajectory starting on the  $x$ -axis above  $x = 2$  will tend to  $x = 2$  from above and any trajectory starting on the  $x$ -axis between  $0$  and  $2$  will tend to  $x = 2$  from below. On the  $y$ -axis  $x' = 0$  and  $y$  is increasing for  $y < 3$  and decreasing for  $y > 3$ . Thus any trajectory starting on the  $y$ -axis above  $y = 3$  will tend to  $y = 3$  from above and any trajectory starting on the  $y$ -axis below  $y = 0$  and  $y > 3$  will tend to  $y = 3$  from below. The whole phase portrait is illustrated in figure 11.

**Remark.** Of course, not always we can determine the whole behaviour

of the system as in the cases above. Especially it is hard to find out if periodic solutions are present or not. Anyhow analysis of the signs of  $x$  and  $y$  and the local behaviour around equilibria give a lot of information about what types of phase portraits are actual.

### Other examples

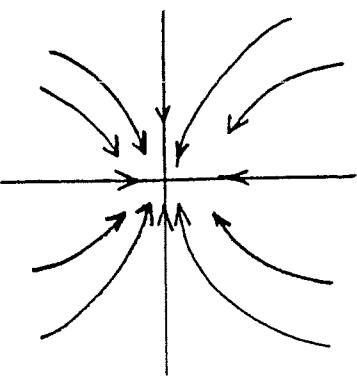
The last figures demonstrate some examples of phase portrait for some systems connected with technical applications. As You see some of them have periodic solutions in the phase portrait.

# Phase portraits of two-dimensional linear systems.

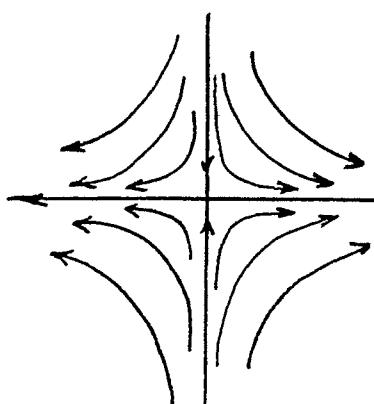
1

Local phase portrait around  $(0,0)$

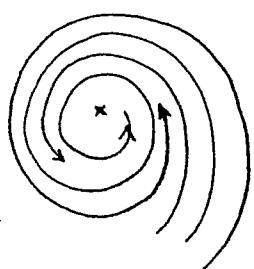
Sinks



Real negative eigenvalues



re positive eigenvalue,  
re negative

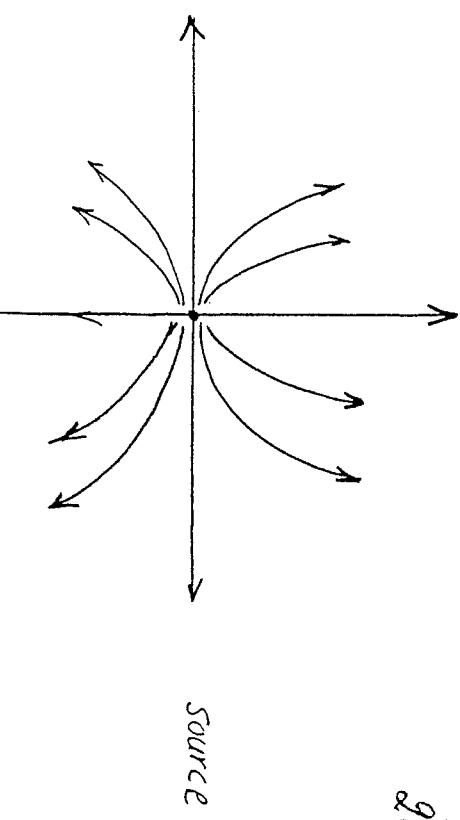


Complex eigenvalues,  
Negative real parts,

Saddle

Local behaviour around equilibria:

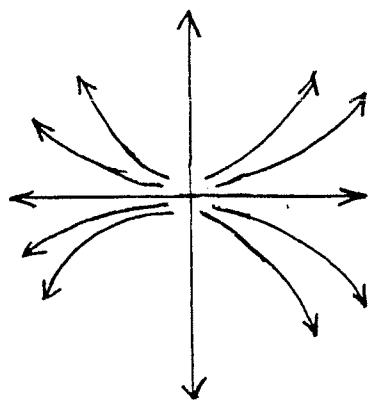
2d



Source

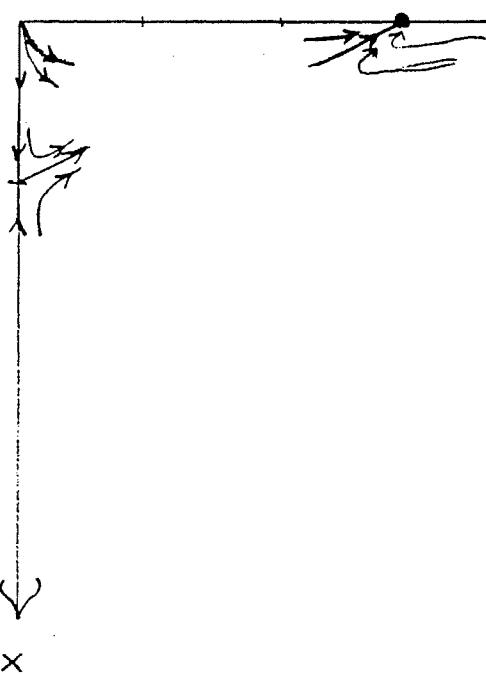
2a

Real positive eigenvalues



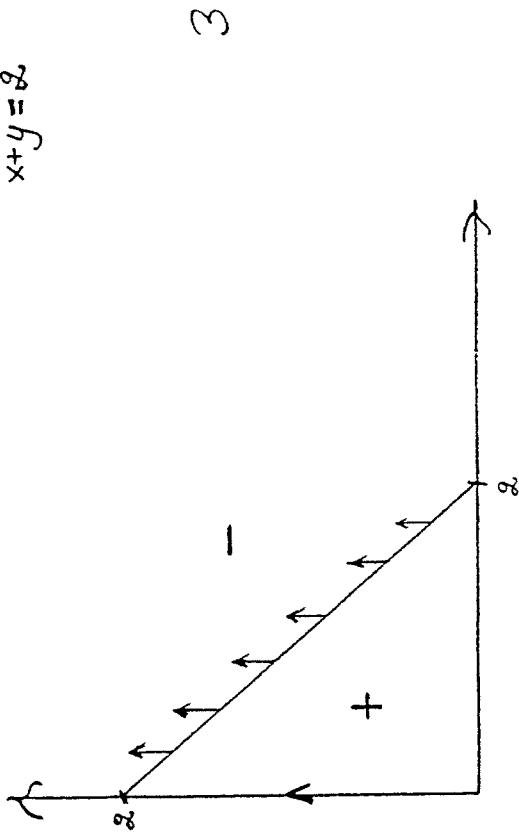
Sources

Complex eigenvalues,  
Positive real parts



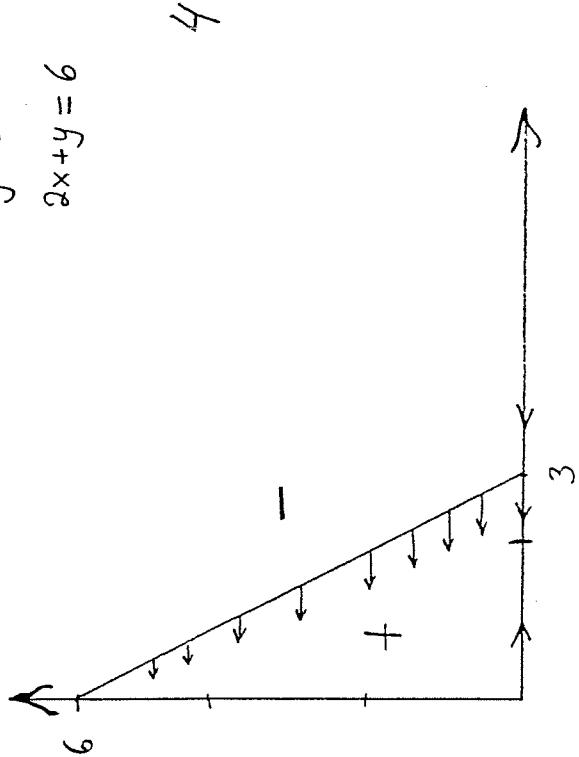
$$\begin{aligned}\dot{x} = 0 \\ \Leftrightarrow x = 0 \\ x + y = 2\end{aligned}$$

The sign of  $\dot{x}$

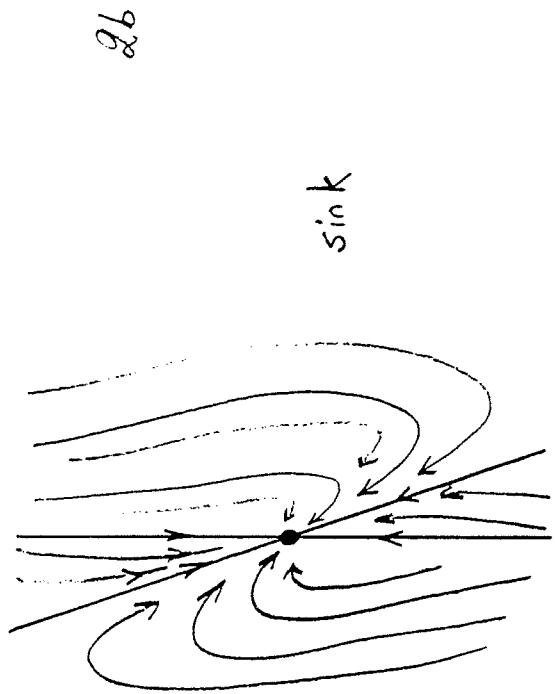


$$\begin{aligned}\dot{y} = 0 \\ \Leftrightarrow y = 0 \\ 2x + y = 6\end{aligned}$$

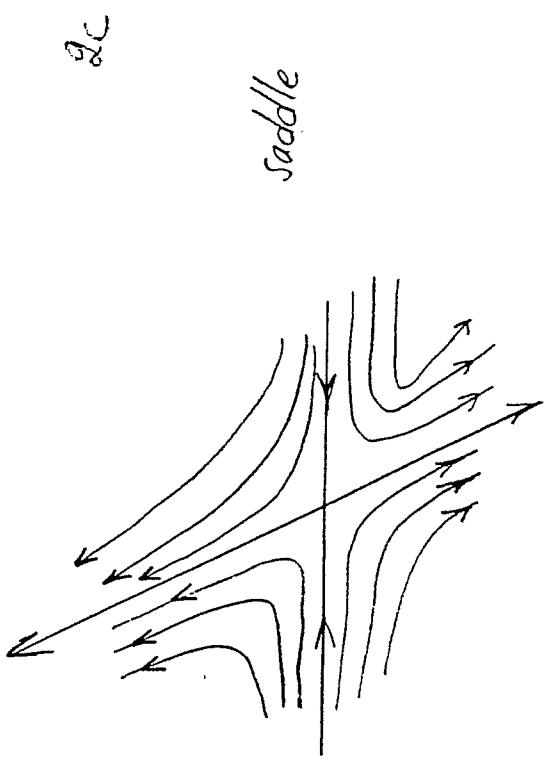
The sign of  $\dot{y}$



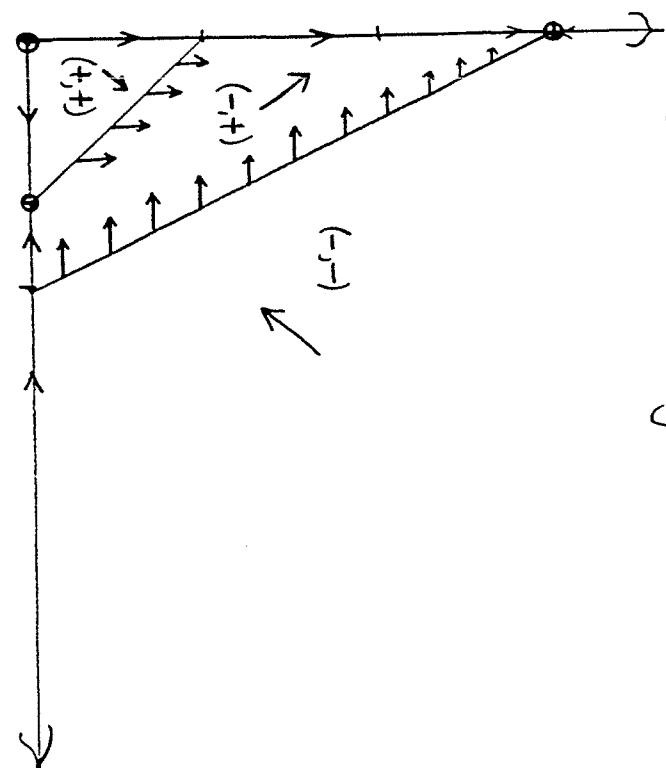
Local phase portrait around  $(0, 6)$



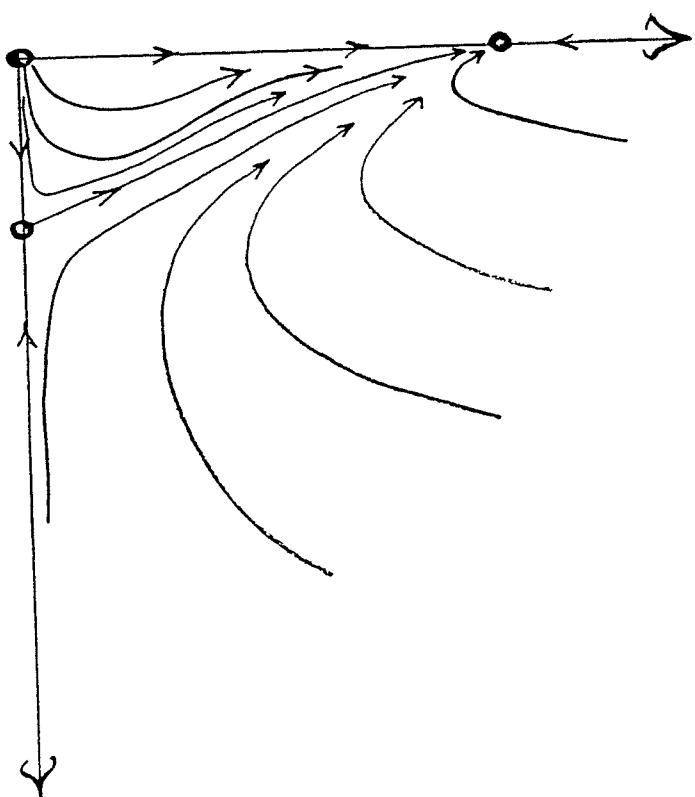
Local phase portrait around  $(2, 0)$



Sign of  $\dot{x}$  and  $\dot{y}$

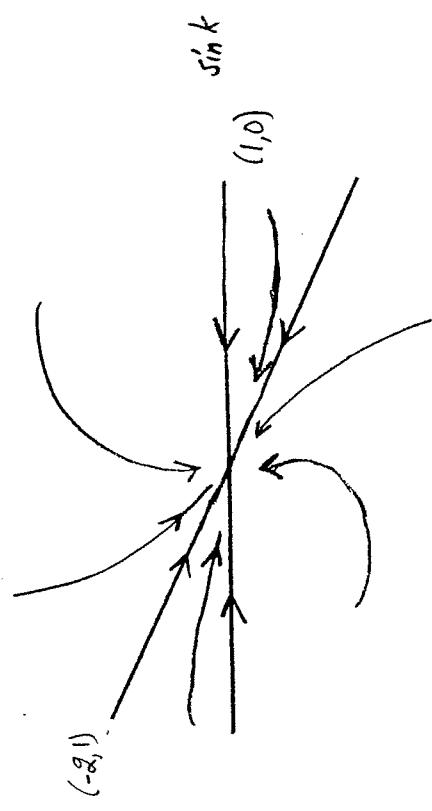


5

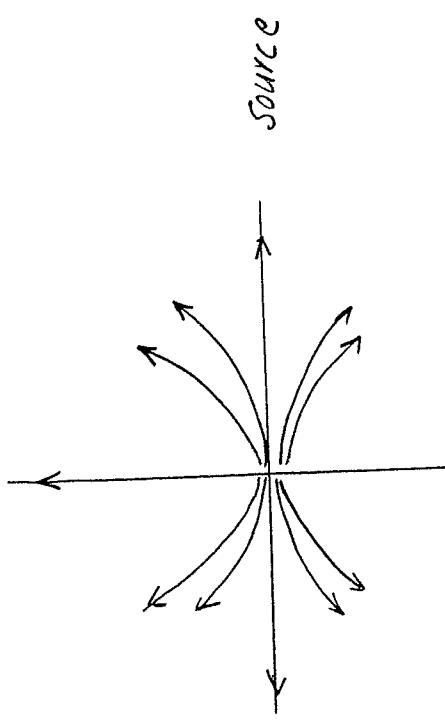


6

Local phase portrait around  $(2, 0)$

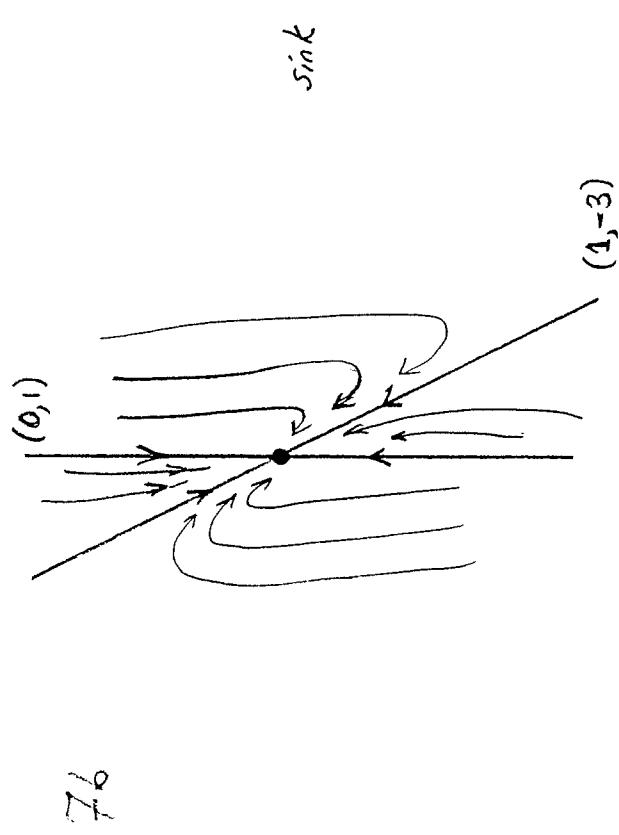


Local phase portrait around  $(0, 0)$



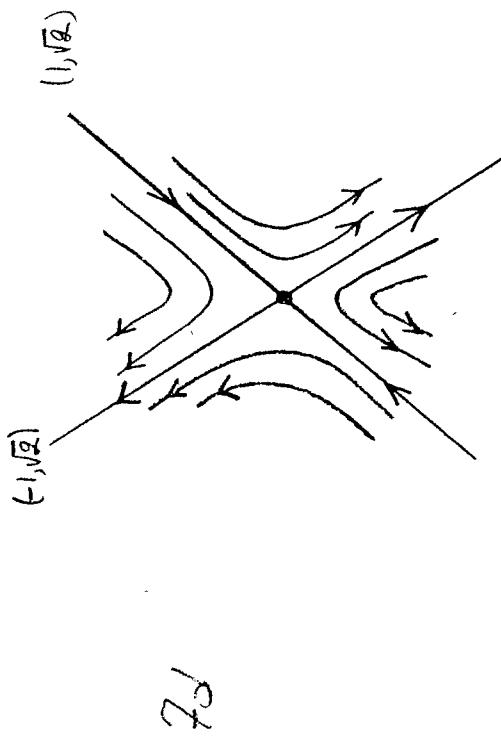
$\mathcal{T}_A$

Local phase portrait around  $(0, 3)$



$\mathcal{T}_B$

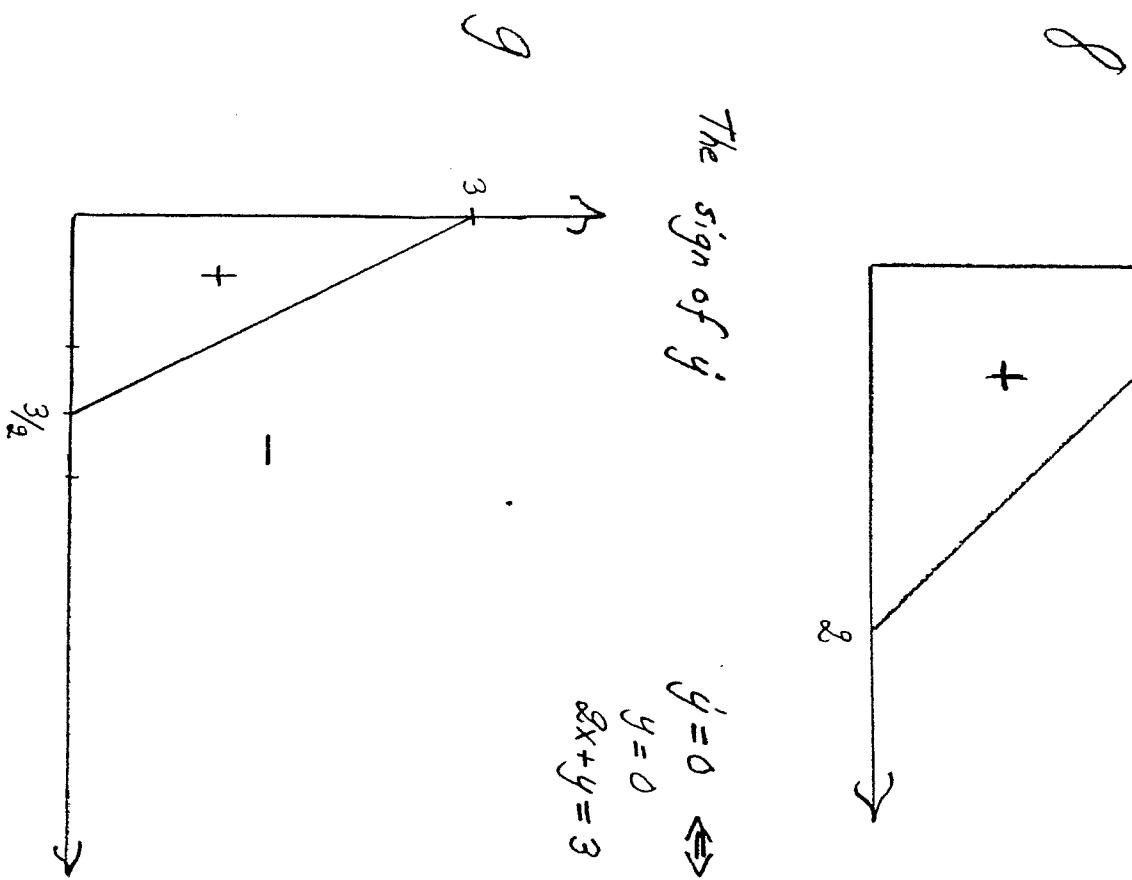
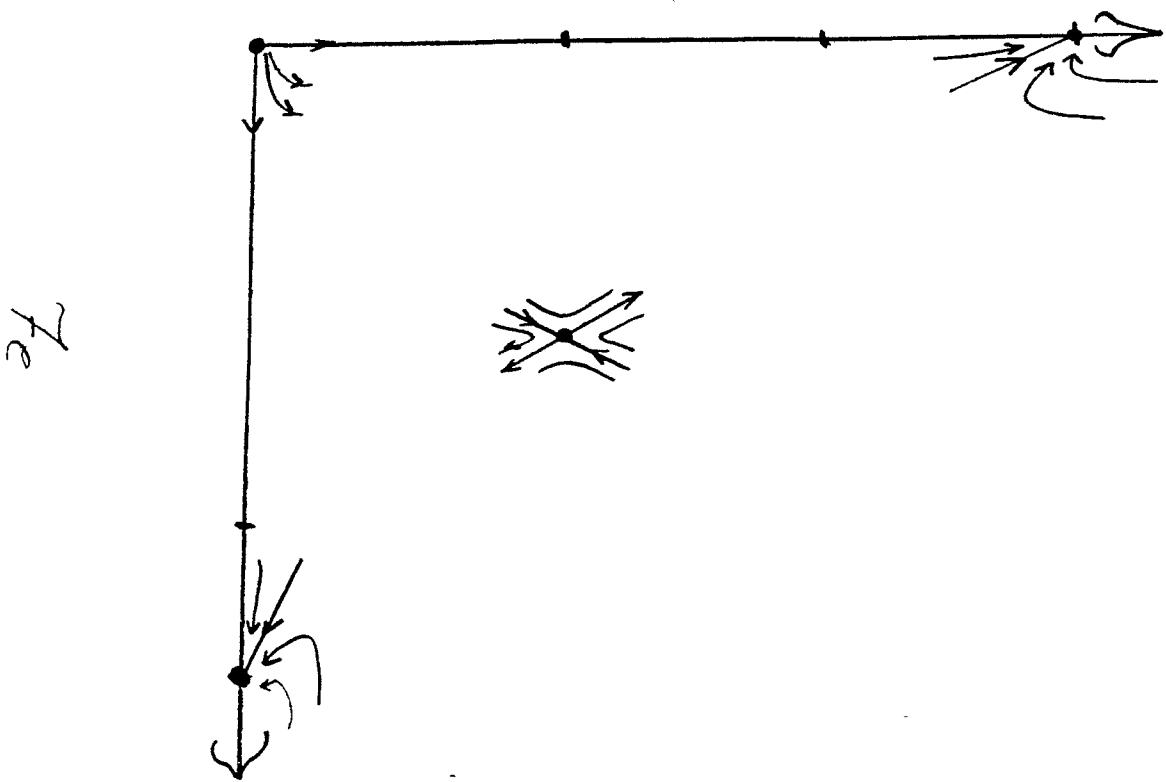
Local phase portrait around  $(1, 1)$



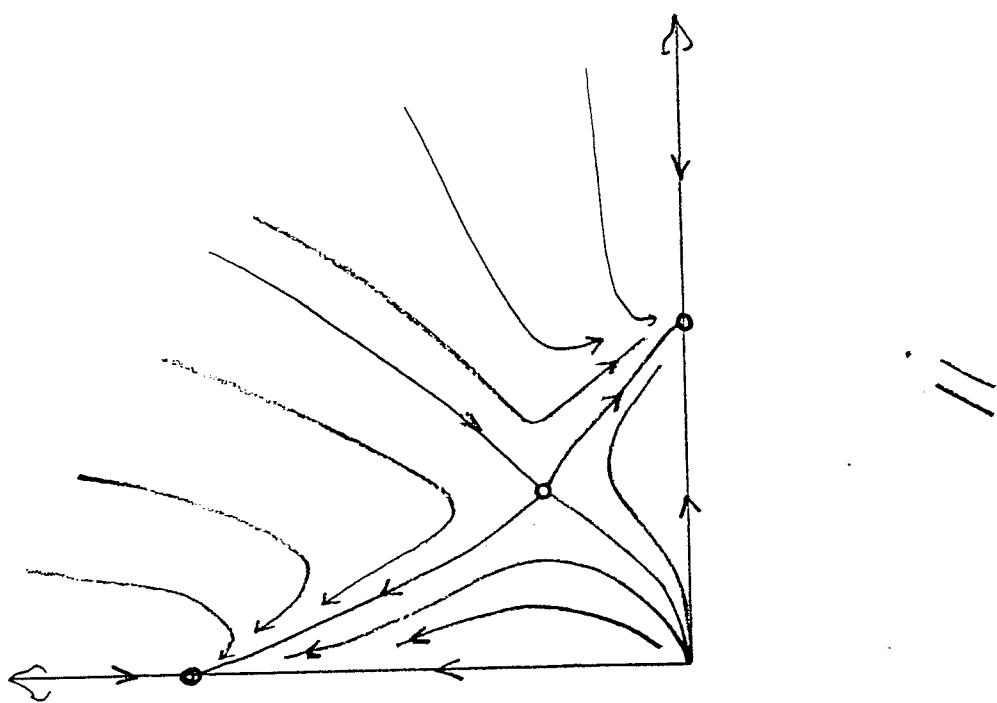
$\mathcal{T}_D$

Local phase portrait around  $(1, 1)$

Local behaviour around equilibria



$$\begin{aligned}
 x = 0 &\Leftrightarrow \\
 x = 0 &\vee \\
 x + y = 2 &
 \end{aligned}$$



Behaviour of  $x', y'$

